10.2.2.1: Stream Function in a Three Dimensions

Pure three dimensional stream functions exist physically but at present there is no known way to represent them mathematically. One of the ways that was suggested by Yih in 1957 suggested using two stream functions to represent the three dimensional flow. The only exception is a stream function for three dimensional flow exists but only for axisymmetric flow i.e the flow properties remains constant in one of the direction (say z axis).

Advance Material

The three dimensional representation is based on the fact the continuity equation must be satisfied. In this case it will be discussed only for incompressible flow. The \( \nabla \cdot \mathbf{U} = 0 \) and vector identity of \( \nabla \cdot \nabla \cdot \mathbf{U} = 0 \) where in this case \( \mathbf{U} \) is any vector. As opposed to two dimensional case, the stream function is defined as a vector function as

\[
\mathbf{B} = \psi \, \nabla \xi \tag{65}
\]

The idea behind this definition is to build stream function based on two scalar functions one provide the "direction" and one provides the the magnitude. In that case, the velocity (to satisfy the continuity equation)

\[
\mathbf{U} = \boldsymbol{\nabla} \times (\psi \, \nabla \chi) \tag{66}
\]

where \( \psi \) and \( \chi \) are scalar functions. Note while \( \psi \) is used here is not the same stream functions that were used in previous cases. The velocity can be obtained by expanding equation (66) to obtained

\[
\mathbf{U} = \nabla \times (\psi \, \nabla \chi)
\]

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\[ \mathbf{U} = \nabla \psi \times \nabla \chi + \psi \left( \nabla \times (\nabla \chi) \right) \tag{67} \]

The second term is zero for any operation of scalar function and hence equation (67) becomes

\[ \mathbf{U} = \nabla \psi \times \nabla \chi \tag{68} \]

These derivations demonstrate that the velocity is orthogonal to two gradient vectors. In another words, the velocity is tangent to the surfaces defined by \( \psi = \text{constant} \) and \( \chi = \text{constant} \). Hence, these functions, \( \psi \) and \( \chi \), are possible stream functions in three dimensions fields. It can be shown that the flow rate is

\[ \dot{Q} = (\psi_2 - \psi_1) (\chi_2 - \chi_1) \tag{69} \]

The answer to the question whether this method is useful and effective is that in some limited situations it could help. In fact, very few research papers deals this method and currently there is not analytical alternative. Hence, this method will not be expanded here.

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