5.6: Discrete -Time Fourier Transform (DTFT)

Learning Objectives

- Discussion of Discrete-time Fourier Transforms.
- Topics include comparison with analog transforms and discussion of Parseval's theorem.

The Fourier transform of the discrete-time signal $s(n)$ is defined to be

$$S(e^{i2\pi f}) = \sum_{n=-\infty}^{\infty} s(n)e^{-i2\pi fn}$$

Frequency here has no units. As should be expected, this definition is linear, with the transform of a sum of signals equaling the sum of their transforms. Real-valued signals have conjugate-symmetric spectra:

$$S(e^{-i2\pi f}) = \overline{S(e^{i2\pi f})}$$

Exercise \(\PageIndex{1}\)

A special property of the discrete-time Fourier transform is that it is periodic with period one:

$$S(e^{i2\pi (f+1)}) = S(e^{i2\pi f})$$

Derive this property from the definition of the DTFT.

Solution

$$S(e^{i2\pi f+1)}) = \sum_{n=-\infty}^{\infty} s(n)e^{-(i2\pi fn)}$$

$$S(e^{i2\pi (f+1)}) = \sum_{n=-\infty}^{\infty} s(n)e^{-(i2\pi fn)}$$
\[ S(e^{i2\pi (f+1)})=\sum_{n=-\infty }^{\infty }s(n)e^{-(i2\pi fn)} \]

Because of this periodicity, we need only plot the spectrum over one period to understand completely the spectrum's structure; typically, we plot the spectrum over the frequency range

\[ \left [ -\frac{1}{2}, \frac{1}{2} \right ] \]

When the signal is real-valued, we can further simplify our plotting chores by showing the spectrum only over

\[ \left [ 0, \frac{1}{2} \right ] \]

the spectrum at negative frequencies can be derived from positive-frequency spectral values.

When we obtain the discrete-time signal via sampling an analog signal, the Nyquist frequency corresponds to the discrete-time frequency 1/2. To show this, note that a sinusoid having a frequency equal to the Nyquist frequency 1/2Ts has a sampled waveform that equals

\[ \cos \left ( \frac{2\pi }{2T_{s}}nT_{s} \right )=\cos (\pi n)=(-1)^{n} \]

The exponential in the DTFT at frequency 1/2 equals

\[ e^{\frac{-i2\pi n}{2}}=e^{-(i\pi n)}=(-1)^{n} \]

meaning that discrete-time frequency equals analog frequency multiplied by the sampling interval

\[ f_{D}=f_{A}T_{s} \]

\( f_{D} \) and \( f_{A} \) represent discrete-time and analog frequency variables, respectively. The aliasing figure provides another way of deriving this result. As the duration of each pulse in the periodic sampling signal \( p_{Ts}(t) \) narrows, the amplitudes of the signal's spectral repetitions, which are governed by the Fourier series coefficients of \( p_{Ts}(t) \), become increasingly equal. Examination of the periodic pulse signal reveals that as \( \Delta \) decreases, the value of \( c_{0} \), the largest Fourier coefficient, decreases to zero:

\[ \left | c_{0} \right |=\frac{A}{T_{s}} \]

Thus, to maintain a mathematically viable Sampling Theorem, the amplitude \( A \) must increase as 1/\( \Delta \), becoming infinitely large as the pulse duration decreases. Practical systems use a small value of \( \Delta \), say 0.1Ts and use amplifiers to rescale the signal. Thus, the sampled signal's spectrum becomes periodic with period 1/Ts. Thus, the Nyquist frequency 1/2Ts corresponds to the frequency 1/2.

Example \( \PageIndex{1} \):

Let's compute the discrete-time Fourier transform of the exponentially decaying sequence

\[ s(n)=a^{n}u(n) \]
where \( u(n) \) is the unit-step sequence. Simply plugging the signal's expression into the Fourier transform formula:

\[
S(e^{i2\pi f}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-(i2\pi fn)}
\]

This sum is a special case of the geometric series.

\[
\sum_{n=0}^{\infty} = \forall \alpha , |\alpha| < 1: \frac{1}{1-\alpha}
\]

Thus, as long as \(|a|<1\), we have our Fourier transform.

\[
S(e^{i2\pi f}) = \frac{1}{1-ae^{-(i2\pi f)}}
\]

Using Euler's relation, we can express the magnitude and phase of this spectrum.

\[
|S(e^{i2\pi f})| = \frac{1}{\sqrt{(1-a\cos(2\pi f))^2 + a^2\sin^2(2\pi f)}}
\]

\[
\angle(S(e^{i2\pi f})) = -\tan^{-1}\left(\frac{a\sin 2\pi f}{1-a\cos (2\pi f)}\right)
\]

No matter what value of \(a\) we choose, the above formulae clearly demonstrate the periodic nature of the spectra of discrete-time signals. Fig. 5.6.1 below shows indeed that the spectrum is a periodic function. We need only consider the spectrum between \(-\frac{1}{2}\) and \(\frac{1}{2}\) to unambiguously define it. When \(a>0\), we have a lowpass spectrum—the spectrum diminishes as frequency increases from 0 to \(\frac{1}{2}\)—with increasing \(a\) leading to a greater low frequency content; for \(a<0\), we have a highpass spectrum as shown in Fig. 5.6.2 below.

![Spectrum](https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Book%3A_Electrical_Engineering_(Johnson)/05%3A_Digital_Si…)

**Fig. 5.6.1** The spectrum of the exponential signal (\(a = 0.5\)) is shown over the frequency range \([-2,2]\), clearly demonstrating the periodicity of all discrete-time spectra. The angle has units of degrees.
Fig. 5.6.1 The spectra of several exponential signals are shown. What is the apparent relationship between the spectra for $a=0.5$ and $a=-0.5$?

Example (PageIndex{1}):

Analogous to the analog pulse signal, let's find the spectrum of the length-$N$ pulse sequence.

$$s(n) = \begin{cases} 1 & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of this sequence has the form of a truncated geometric series.

$$S(e^{i2\pi f}) = \sum_{n=0}^{N-1} e^{-(i2\pi fn)}$$

For the so-called finite geometric series, we know that

$$\sum_{n=n_0}^{N+n_0-1} \alpha^n = \alpha^{n_0} \frac{1-\alpha^N}{1-\alpha}$$

for all values of $\alpha$.

Exercise (PageIndex{1})

Derive this formula for the finite geometric series sum. The "trick" is to consider the difference between the series' sum and the sum of the series multiplied by $\alpha$.

Solution

$$\alpha \sum_{n=n_0}^{N+n_0-1} \alpha^n = \alpha^{n_0} \frac{1-\alpha^N}{1-\alpha}$$

which, after manipulation, yields the geometric sum formula.

Applying this result yields to Fig. 5.6.3,
\[ S(e^{i2\pi f}) = \frac{1 - e^{-(i2\pi fN)}}{1 - e^{-(i2\pi f)}} \]

\[ S(e^{i2\pi f}) = e^{-i\pi f(N-1)} \frac{\sin(\pi fN)}{\sin(\pi f)} \]

The ratio of sine functions has the generic form of

\[ \frac{\sin(Nx)}{\sin(x)} \]

which is known as the **discrete-time sinc function** \( dsinc(x) \). Thus, our transform can be concisely expressed as

\[ S(e^{i2\pi f}) = e^{-i\pi f(N-1)} dsinc(\pi f) \]

The discrete-time pulse's spectrum contains many ripples, the number of which increase with \( N \), the pulse's duration.

**Fig. 5.6.3** The spectrum of a length-ten pulse is shown. Can you explain the rather complicated appearance of the phase?

The inverse discrete-time Fourier transform is easily derived from the following relationship:

\[
\int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-(i2\pi fm)} e^{i2\pi fn} df = \begin{cases} 1 & \text{ if } m=n \\ 0 & \text{ if } m \neq n \end{cases}
\]

\[ \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-(i2\pi fm)} e^{i2\pi fn} df = \delta(m-n) \]

Therefore, we find that

\[
\int_{\frac{-1}{2}}^{\frac{1}{2}} S(e^{i2\pi f}) e^{i2\pi fn} df = \sum_{m,n} \begin{cases} 1 & \text{ if } m=n \\ 0 & \text{ if } m \neq n \end{cases}
\]

\[ \int_{\frac{-1}{2}}^{\frac{1}{2}} S(e^{i2\pi f}) e^{i2\pi fn} df = \delta(m-n) \]

\[ \int_{\frac{-1}{2}}^{\frac{1}{2}} S(e^{i2\pi f}) e^{i2\pi fn} df = s(n) \]

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The Fourier transform pairs in discrete-time are

\[ \mathcal{S}(e^{i2\pi f}) = \sum_{n=-\infty}^{\infty} s(n) e^{-i2\pi fn} \]
\[ s(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{S}(e^{i2\pi f}) e^{i2\pi fn} df \]

The properties of the discrete-time Fourier transform mirror those of the analog Fourier transform. The DTFT properties table below shows similarities and differences. One important common property is Parseval's Theorem.

### Summary Table of DTFT Properties

<table>
<thead>
<tr>
<th>Sequence Domain</th>
<th>Frequency Domain</th>
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</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>[ a[n] + b[n] ]</td>
</tr>
<tr>
<td>Conjugate Symmetry</td>
<td>[ \mathcal{S}\left( e^{-i\omega} \right) = \mathcal{S}(e^{i\omega}) ]</td>
</tr>
<tr>
<td>Even Symmetry</td>
<td>[ s[n] = s[-n] ]</td>
</tr>
<tr>
<td>Odd Symmetry</td>
<td>[ s[n] = -s[-n] ]</td>
</tr>
<tr>
<td>Time Delay</td>
<td>[ s[n-k] ]</td>
</tr>
<tr>
<td>Multiplication by n</td>
<td>[ s[n] \rightarrow s[n]e^{i\theta n} ]</td>
</tr>
<tr>
<td>Sum</td>
<td>[ \sum_{n=-\infty}^{\infty} ]</td>
</tr>
<tr>
<td>Value at Origin</td>
<td>[ s(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}}</td>
</tr>
<tr>
<td>Parseval's Theorem</td>
<td>[ \sum_{n=-\infty}^{\infty}</td>
</tr>
<tr>
<td>Complete Modulation</td>
<td>[ e^{i2\pi n \omega} \rightarrow S(e^{i2\pi f + \omega}) ]</td>
</tr>
<tr>
<td>Amplitude Modulation</td>
<td>[</td>
</tr>
</tbody>
</table>

To show this important property, we simply substitute the Fourier transform expression into the frequency-domain expression for power.

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathcal{S}(e^{i2\pi f})|^{2} df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \sum_{n=-\infty}^{\infty} s(n) e^{-i2\pi fn} \right) \overline{s(n)} e^{i2\pi fm} df \]

Using the orthogonality relation, the integral equals \( \delta(m-n) \), where \( \delta(n) \) is the unit sample. Thus, the double sum collapses into a single sum because nonzero values occur only when \( n=m \), giving Parseval's Theorem as a result. We term

\[ \sum_{n=-\infty}^{\infty} |s(n)|^{2} \]

the energy in the discrete-time signal \( s(n) \) in spite of the fact that discrete-time signals don't consume (or produce for that matter) energy. This terminology is a carry-over from the analog world.
Exercise

Suppose we obtained our discrete-time signal from values of the product $s(t)p_{Ts}(t)$, where the duration of the component pulses in $p_{Ts}(t)$ is $\Delta$. How is the discrete-time signal energy related to the total energy contained in $s(t)$? Assume the signal is bandlimited and that the sampling rate was chosen appropriate to the Sampling Theorem's conditions.

**Solution**

If the sampling frequency exceeds the Nyquist frequency, the spectrum of the samples equals the analog spectrum, but over the normalized analog frequency $fT$. Thus, the energy in the sampled signal equals the original signal's energy multiplied by $T$.

**Contributor**

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