4.9: Linear Time Invariant Systems

Learning Objectives

• Application of periodic input to linear, time-invariant systems.

When we apply a periodic input to a linear, time-invariant system, the output is periodic and has Fourier series coefficients equal to the product of the system's frequency response and the input's Fourier coefficients (Filtering Periodic Signals). The way we derived the spectrum of non-periodic signal from periodic ones makes it clear that the same kind of result works when the input is not periodic.

Note

If \(x(t)\) serves as the input to a linear, time-invariant system having frequency response \(H(f)\), the spectrum of the output is \(X(f)H(f)\).

Example 1:

Let's use this frequency-domain input-output relationship for linear, time-invariant systems to find a formula for the RC-circuit's response to a pulse input. We have expressions for the input's spectrum and the system's frequency response.

\[
P(f) = e^{-(i\pi f \Delta)} \left( \frac{\sin(\pi f \Delta)}{\pi f} \right)
\]

\[
H(f) = \frac{1}{1 + i2\pi f RC}
\]

The spectrum of the output is then:

\[
X(f)H(f) = \left( \frac{\sin(\pi f \Delta)}{\pi f} \right) \left( \frac{1}{1 + i2\pi f RC} \right)
\]
Thus, the output’s Fourier transform equals

$$Y(f) = e^{-i\pi f \Delta} \frac{\sin (\pi f \Delta)}{\pi f} \frac{1}{1+i2\pi f RC} \nonumber$$

You won’t find this Fourier transform in our table, and the required integral is difficult to evaluate as the expression stands. This situation requires cleverness and an understanding of the Fourier transform’s properties. In particular, recall Euler’s relation for the sinusoidal term and note the fact that multiplication by a complex exponential in the frequency domain amounts to a time delay. Let’s momentarily make the expression for $Y(f)$ more complicated.

$$\left[ e^{-i\pi f \Delta} \frac{\sin (\pi f \Delta)}{\pi f} = e^{-i\pi f \Delta} \frac{e^{i\pi f \Delta} - e^{-i\pi f \Delta}}{i2\pi f} \right] \nonumber$$

$$\left[ e^{-i\pi f \Delta} \frac{\sin (\pi f \Delta)}{\pi f} = \frac{1}{i2\pi f} \left( 1 - e^{-i\pi f \Delta} \right) \right] \nonumber$$

Consequently,

$$Y(f) = \frac{1}{i2\pi f} \left( 1 - e^{-i\pi f \Delta} \right) \frac{1}{1+i2\pi f RC} \nonumber$$

The table of Fourier transform properties suggests thinking about this expression as a product of terms.

- Multiplication by $1/i2\pi f$ means integration.
- Multiplication by the complex exponential $\left[ e^{-(i2\pi f \Delta)} \right]$ means delay by $\Delta$ seconds in the time domain.
- The term $\left[ 1 - e^{-(i2\pi f \Delta)} \right]$ means, in the time domain, subtract the time-delayed signal from its original.
- The inverse transform of the frequency response is $\left[ \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \right] \nonumber$

We can translate each of these frequency-domain products into time-domain operations in any order we like because the order in which multiplications occur doesn’t affect the result. Let’s start with the product of $1/i2\pi f$ (integration in the time domain) and the transfer function:

$$\left[ \frac{1}{i2\pi f} \frac{1}{1+i2\pi f RC} \right] \leftrightarrow \left( 1 - e^{-\frac{t}{RC}} \right) u(t) \nonumber$$

The middle term in the expression for $Y(f)$ consists of the difference of two terms: the constant 1 and the complex exponential $\left[ e^{-(i2\pi f \Delta)} \right]$.

Because of the Fourier transform’s linearity, we simply subtract the results.

$$Y(f) \left[ \frac{1}{i2\pi f} \frac{1}{1+i2\pi f RC} \right] \left[ 1 - e^{-\frac{t}{RC}} \right] u(t) \nonumber$$

Note that in delaying the signal we carefully included the unit step. The second term in this result does not begin until $t = \Delta$. Thus, the waveforms shown in the Filtering Periodic Signals example mentioned above are exponentials. We say that the time constant of an exponentially decaying signal equals the time it takes to decrease by $1/e$ of its original value. Thus, the time-constant of the rising and falling portions of the output equal the product of the circuit’s resistance and capacitance.
Exercise \(\PageIndex{1}\))

Derive the filter’s output by considering the terms in the above \(Y(f)\) equation in the order given. Integrate last rather than first. You should get the same answer.

Solution

The inverse transform of the frequency response is \[
\frac{1}{RC}e^{-\frac{t}{RC}}u(t) \nonumber
\]

Multiplying the frequency response by \[1-e^{-(i2\pi f \Delta)} \nonumber\] means subtract from the original signal its time-delayed version. Delaying the frequency response’s time-domain version by \(\Delta\) results in \[\frac{1}{RC}e^{-\frac{(t-\Delta)}{RC}}u(t-\Delta) \nonumber\]

Subtracting from the undelayed signal yields

\[
\frac{1}{RC}e^{-\frac{t}{RC}}u(t) - \frac{1}{RC}e^{-\frac{(t-\Delta)}{RC}}u(t-\Delta) \nonumber
\]

Now we integrate this sum. Because the integral of a sum equals the sum of the component integrals (integration is linear), we can consider each separately. Because integration and signal-delay are linear, the integral of a delayed signal equals the delayed version of the integral. The integral is provided in the above example.

In this example, we used the table extensively to find the inverse Fourier transform, relying mostly on what multiplication by certain factors, like \(\frac{1}{i2\pi f}\) and \(e^{-(i2\pi f \Delta)} \nonumber\) meant. We essentially treated multiplication by these factors as if they were transfer functions of some fictitious circuit. The transfer function \(\frac{1}{i2\pi f}\) corresponded to a circuit that integrated, and \(e^{-(i2\pi f \Delta)} \nonumber\) to one that delayed. We even implicitly interpreted the circuit’s transfer function as the input’s spectrum! This approach to finding inverse transforms -- breaking down a complicated expression into products and sums of simple components -- is the engineer’s way of breaking down the problem into several subproblems that are much easier to solve and then gluing the results together. Along the way we may make the system serve as the input, but in the rule

\[
[Y(f)=X(f)H(f) \nonumber]
\]

which term is the input and which is the transfer function is merely a notational matter (we labeled one factor with an \(X\) and the other with an \(H\)).

Transfer Functions

The notion of a transfer function applies well beyond linear circuits. Although we don't have all we need to demonstrate the result as yet, all linear, time-invariant systems have a frequency-domain input-output relation given by the product of the input's Fourier transform and the system's transfer function. Thus, linear circuits are a special case of linear, time-invariant systems. As we tackle more sophisticated problems in transmitting, manipulating, and receiving information, we will assume linear systems having certain properties (transfer functions) without worrying about what circuit has the desired property. At this point, you may be concerned that this approach is glib, and rightly so. Later we'll show that by involving software that we really don’t need to be concerned about constructing a transfer function from circuit elements and op-amps.
Commutative Transfer Functions

Another interesting notion arises from the commutative property of multiplication (exploited in the above example). We can rather arbitrarily choose an order in which to apply each product. Consider a cascade of two linear, time-invariant systems. Because the Fourier transform of the first system's output is:

\[X(f)H_{\{1\}}(f)\]

and it serves as the second system's input, the cascade's output spectrum is:

\[X(f)H_{\{1\}}(f)H_{\{2\}}(f)\]

Because this product also equals

\[X(f)H_{\{2\}}(f)H_{\{1\}}(f)\]

the cascade having the linear systems in the opposite order yields the same result. Furthermore, the cascade acts like a single linear system, having transfer function

\[H_{\{1\}}(f)H_{\{2\}}(f)\]

This result applies to other configurations of linear, time-invariant systems as well; see this Frequency Domain Problem. Engineers exploit this property by determining what transfer function they want, then breaking it down into components arranged according to standard configurations. Using the fact that op-amp circuits can be connected in cascade with the transfer function equaling the product of its component's transfer function (see this analog signal processing problem), we find a ready way of realizing designs. We now understand why op-amp implementations of transfer functions are so important.