6.7: The advection-diffusion equation for a scalar concentration

The composition-dependent equations of state all require adding a new variable to the equations of motion: \(q\) in the case of moist air, \(S\) in the case of seawater. Closure then requires that one more equation be added to account for the new variable. In the case of moist air, the new equation must account for the complex thermodynamics and chemistry of water vapor, and we will not go into it here (for a detailed discussion see Curry and Webster 1998). In the case of salinity, the solution is much simpler.

Salinity \(S\) is the mass of salt, in grams, per unit mass of salt water, in kilograms. The mass of salt in a fluid parcel is therefore given by \(\int_V \rho S \, dV\). That mass of salt can change only by means of exchanges with the environment, represented by a salt flux \(\vec{J}_S\):

\[
\frac{D}{D t} \int_V \rho S \, dV = -\int_{A} \vec{J}_S \cdot \hat{n} \, dA = -\int_V \nabla \cdot \vec{J}_S \, dV,
\]

where the divergence theorem has been used for the final step. This Lagrangian statement can be converted to Eulerian form in the usual way, resulting in

\[
\rho \frac{D S}{D t} = -\nabla \cdot \vec{J}_S.
\]

The salt flux is well approximated by Fick’s law:

\[
\vec{J}_S = -\rho \kappa_S \nabla S,
\]

where \(\kappa_S\) is the molecular diffusivity of salt, with typical value \(10^{-9} \text{ m}^2 \text{ s}^{-1}\). Neglecting small variations in \(\rho\), we have a diffusion equation for salinity:

\[
\frac{D S}{D t} = \kappa_S \nabla^2 S. \tag{1}
\]