6.2: Deflections of Circular Plates

The governing equation (6.1.9) still holds but the Laplace operator \( \nabla^2 \) should now be defined in the polar coordinate system \((r, \theta)\):

\[
\nabla^2 w = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \tag{7.15}
\]

In the circular plate subjected to axi-symmetric loading \((p = p(r))\), the third term in Equation \ref{7.15} vanishes and the Laplace operator can be put in the form

\[
\nabla^2 w = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right)
\]

With the above definition, the plate bending equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) \right) = \frac{p(r)}{D}
\]

and the solution is obtained by four successive integration

\[
w(r) = \int \int \int \int \frac{1}{r} \int r \int \frac{1}{r} \frac{rp(r)}{D} \, dr \, dr \, dr \, dr
\]

Assuming a uniform loading of the intensity \((p_o)\), the above integration can be easily performed to give

\[
w(r) = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4 + \frac{p_or^4}{64D} \tag{7.19}
\]

As an illustration, consider clamped boundary conditions:

\[
\text{at } r = R \quad w = 0 \quad \text{and} \quad \frac{dw}{dr} = 0
\]
\[ \text{at } r = 0 \quad \frac{dw}{dr} = 0 \quad \text{and} \quad \bar{V}_r = 0 \]

where the shear force (per unit length), acting on a plate element at a distance \( r \) is

\[ \bar{V}_r = - \frac{1}{2\pi r} \int_0^r p_o 2\pi r \, dr = - \frac{p_o r}{2} \]

The two terms in Equation \ref{7.19} involving logarithms tend to infinity at \( r \rightarrow 0 \). Therefore, in order for the solution to give finite values of deflections at the center, \( (C_1 = C_3 = 0) \). Now, the expression for the slope is

\[ \frac{dw}{dr} = 2C_2 r + \frac{p_o r^3}{18D} \]

Now, the boundary conditions at \( r = 0 \) are satisfied identically. From two boundary conditions at \( r = R \), one finds the integration constants

\[ [C_2 = - \frac{p_o R^2}{32D}, \quad C_4 = \frac{p_o R^4}{64D}] \]

The final form of the solution for the plate deflection is

\[ w(r) = \frac{p_o R^4}{64D} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^2 \]

For comparison, the solution for the simply supported plate will be derived. The boundary conditions are mixed so the moment-curvature relation must be used

\[ M_r = D \left[ \frac{d^2w}{dr^2} + \nu \frac{1}{r} \frac{dw}{dr} \right] \quad \text{and} \quad M_{\theta} = D \left[ \frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2w}{dr^2} \right] \]

where the definition of moments in the cylindrical coordinate system was used. At the plate edge

\[ w = 0 \quad \text{and} \quad M_r = 0 \quad \text{at} \quad r = R \]

From Equations \ref{7.19}, \ref{7.25a} - \ref{7.25b} and \ref{7.26}, the system of two algebraic equations for \( C_2 \) and \( C_4 \) is obtained, where solution is

\[ [C_2 = - \frac{p_o R^2}{32D} \frac{3 + \nu}{1 + \nu}, \quad C_4 = \frac{p_o R^4}{64D} \frac{5 + \nu}{1 + \nu}] \]

The formula for the plate deflection is

\[ w(r) = \frac{p R^4}{64D} \left[ \left( \frac{r}{R} \right)^4 - 2 \left( \frac{r}{R} \right)^2 \frac{3 + \nu}{1 + \nu} + \frac{5 + \nu}{1 + \nu} \right] \]

The ratio of the maximum deflection of the simply supported and clamped plate at \( r = 0 \) is

\[ \frac{w_{\text{simplysupported}}}{w_{\text{clamped}}} = \frac{5 + \nu}{1 + \nu} \approx 4 \]

It is interesting that a similar ratio for beams is exactly 5.
Clamped plate is four times stiffer than the simply supported circular plate.

The clamped circular plate can be a prototype of the whole family of similar plates. It is therefore of interest to explore the properties of the above solution further. From Equation \ref{7.24} the radial and circumferential curvatures are:

\[
\kappa_r = -\frac{d^2w}{dr^2} = \frac{p_oR^2}{16D} \left(1 - \frac{3r^2}{R^2}\right)
\]

\[
\kappa_{\theta} = -\frac{1}{r}\frac{dw}{dr} = \frac{p_oR^2}{16D} \left(1 - \frac{r^2}{R^2}\right)
\]

From the constitutive equations, the radial and circumferential bending moments are

\[
M_r = D\left[\kappa_r + \nu \kappa_{\theta}\right] + \frac{p_oR^2}{16} \left[ (1 + \nu) - (3 + 3\nu) \left(\frac{r}{R}\right)^2 \right]
\]

\[
M_{\theta} = D\left[\kappa_{\theta} + \nu \kappa_r\right] + \frac{p_oR^2}{16} \left[ (1 + \nu) - (1 + 3\nu) \left(\frac{r}{R}\right)^2 \right]
\]

At the plate center, by symmetry

\[
M_r = M_{\theta} = (1 + \nu) \frac{p_oR^2}{16} \quad \text{\ref{7.32}}
\]

Another extreme value occurs at the clamped edge

\[
M_r = \frac{p_oR^2}{8} , \quad M_{\theta} = -\nu \frac{p_oR^2}{8} \text{ at } r = R \quad \text{\ref{7.33}}
\]

By comparing Equations \ref{7.32} and \ref{7.33}, it is seen that the maximum bending moment occurs at the edge \(r = R\). From the stress formula, Equation \(3.7.7\)

\[
|\sigma_{rr}| = \left|\frac{M_rz}{h^3/12}\right|_{z=\frac{h}{2}} = p_o \frac{3}{4}\left(\frac{R}{h}\right)^2
\]

At the same time, the circumferential bending moment at \(r = R\) is

\[
|\sigma_{\theta \theta}| = \left|\frac{M_{\theta}z}{h^3/12}\right| = p_o \frac{1}{4}\left(\frac{R}{h}\right)^2
\]
Figure \(\PageIndex{2}\): Variation of radial and circumferential stresses along the radius of the plate.