3.3: Continuous Time Convolution

Introduction

Convolution, one of the most important concepts in electrical engineering, can be used to determine the output a system produces for a given input signal. It can be shown that a linear time invariant system is completely characterized by its impulse response. The sifting property of the continuous time impulse function tells us that the input signal to a system can be represented as an integral of scaled and shifted impulses and, therefore, as the limit of a sum of scaled and shifted approximate unit impulses. Thus, by linearity, it would seem reasonable to compute of the output signal as the limit of a sum of scaled and shifted unit impulse responses and, therefore, as the integral of a scaled and shifted impulse response. That is exactly what the operation of convolution accomplishes. Hence, convolution can be used to determine a linear time invariant system's output from knowledge of the input and the impulse response.

Convolution and Circular Convolution

Convolution

Operation Definition

Continuous time convolution is an operation on two continuous time signals defined by the integral

\[
(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, d\tau
\]

for all signals \( f, g \) defined on \( \mathbb{R} \). It is important to note that the operation of convolution is commutative,
meaning that
\[ f^{*} g = g^{*} f \]
for all signals \( f \), \( g \) defined on \( \mathbb{R} \). Thus, the convolution operation could have been just as easily stated using the equivalent definition
\[ (f^{*} g)(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \, d\tau \]
for all signals \( f \), \( g \) defined on \( \mathbb{R} \). Convolution has several other important properties not listed here but explained and derived in a later module.

**Definition Motivation**

The above operation definition has been chosen to be particularly useful in the study of linear time invariant systems. In order to see this, consider a linear time invariant system \( H \) with unit impulse response \( h \). Given a system input signal \( x \) we would like to compute the system output signal \( H(x) \). First, we note that the input can be expressed as the convolution
\[ x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \]
by the sifting property of the unit impulse function. Writing this integral as the limit of a summation,
\[ x(t) = \lim_{\Delta \to 0} \sum_{n} x(n \Delta) \delta_{\Delta}(t-n \Delta) \Delta \]
where
\[ \delta_{\Delta}(t) = \begin{cases} 1/\Delta & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases} \]
approximates the properties of \( \delta(t) \). By linearity
\[ H x(t) = \lim_{\Delta \to 0} \sum_{n} x(n \Delta) H \delta_{\Delta}(t-n \Delta) \Delta \]
which evaluated as an integral gives
\[ H x(t) = \int_{-\infty}^{\infty} x(\tau) H \delta(t-\tau) \, d\tau \]
Since \( H \delta(t-\tau) \) is the shifted unit impulse response \( h(t-\tau) \), this gives the result
\[ H x(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau = \text{left}(x^{*} h)(t) \]
Hence, convolution has been defined such that the output of a linear time invariant system is given by the convolution of the system input with the system unit impulse response.
Graphical Intuition

It is often helpful to be able to visualize the computation of a convolution in terms of graphical processes. Consider the convolution of two functions \( f \), \( g \) given by

\[
(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, d\tau = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \, d\tau
\]

The first step in graphically understanding the operation of convolution is to plot each of the functions. Next, one of the functions must be selected, and its plot reflected across the \( \tau=0 \) axis. For each real \( t \), that same function must be shifted left by \( t \). The product of the two resulting plots is then constructed. Finally, the area under the resulting curve is computed.

Example \( \PageIndex{1} \)

Recall that the impulse response for the capacitor voltage in a series RC circuit is given by

\[
h(t) = \frac{1}{RC} e^{-t/RC} u(t),
\]

and consider the response to the input voltage

\[
x(t) = u(t).
\]

We know that the output for this input voltage is given by the convolution of the impulse response with the input signal

\[
y(t) = x(t) * h(t)
\]

We would like to compute this operation by beginning in a way that minimizes the algebraic complexity of the expression. Thus, since \( \langle x(t) = u(t) \rangle \) is the simpler of the two signals, it is desirable to select it for time reversal and shifting. Thus, we would like to compute

\[
y(t) = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\tau/RC} u(\tau) u(t-\tau) \, d\tau
\]

The step functions can be used to further simplify this integral by narrowing the region of integration to the nonzero region of the integrand. Therefore,

\[
y(t) = \int_{0}^{\max\{0, t\}} \frac{1}{RC} e^{-\tau/RC} \, d\tau
\]

Hence, the output is

\[
y(t) = \begin{cases} 
0 & t \leq 0 \\
1-e^{-t/RC} & t > 0
\end{cases}
\]

which can also be written as

\[
y(t) = (1-e^{-t/RC}) u(t).
\]
Circular Convolution

Continuous time circular convolution is an operation on two finite length or periodic continuous time signals defined by the integral

\[(f * g)(t)=\int_{0}^{T} \hat{f}(\tau) \hat{g}(t-\tau) d \tau\]

for all signals \(f\), \(g\) defined on \(\mathbb{R}[0, T]\) where \(\hat{f}\), \(\hat{g}\) are periodic extensions of \(f\) and \(g\). It is important to note that the operation of circular convolution is commutative, meaning that

\[(f * g) = (g * f)\]

for all signals \(f\), \(g\) defined on \(\mathbb{R}[0, T]\). Thus, the circular convolution operation could have been just as easily stated using the equivalent definition

\[(f * g)(t)=\int_{0}^{T} \hat{f}(t-\tau) \hat{g}(\tau) d \tau\]

for all signals \(f\), \(g\) defined on \(\mathbb{R}[0, T]\) where \(\hat{f}\), \(\hat{g}\) are periodic extensions of \(f\) and \(g\). Circular convolution has several other important properties not listed here but explained and derived in a later module.

Alternatively, continuous time circular convolution can be expressed as the sum of two integrals given by

\[(f * g)(t)=\int_{0}^{t} f(\tau) g(t-\tau) d \tau + \int_{t}^{T} f(\tau) g(t-\tau+T) d \tau\]

for all signals \(f\), \(g\) defined on \(\mathbb{R}[0, T]\).

Meaningful examples of computing continuous time circular convolutions in the time domain would involve complicated algebraic manipulations dealing with the wrap around behavior, which would ultimately be more confusing than helpful. Thus, none will be provided in this section. However, continuous time circular convolutions are more easily computed using frequency domain tools as will be shown in the continuous time Fourier series section.

Definition Motivation

The above operation definition has been chosen to be particularly useful in the study of linear time invariant systems. In order to see this, consider a linear time invariant system \(H\) with unit impulse response \(h\). Given a finite or periodic system input signal \(x\) we would like to compute the system output signal \(H(x)\). First, we note that the input can be expressed as the circular convolution

\[x(t)=\int_{0}^{T} \hat{x}(\tau) \hat{\delta}(t-\tau) d \tau\]

by the sifting property of the unit impulse function. Writing this integral as the limit of a summation,

\[x(t)=\lim_{\Delta \to 0} \sum_{n} \hat{x}(n \Delta) \hat{\delta}_{\Delta}(t-n \Delta) \Delta\]

where

\[\hat{\delta}_{\Delta}(t)=\left\{\begin{array}{cc}
1 & \text{if } 0 \leq t < \Delta \\
0 & \text{otherwise}
\end{array}\right\}\]

Meaningful examples of computing continuous time circular convolutions in the time domain would involve complicated algebraic manipulations dealing with the wrap around behavior, which would ultimately be more confusing than helpful. Thus, none will be provided in this section. However, continuous time circular convolutions are more easily computed using frequency domain tools as will be shown in the continuous time Fourier series section.
\[
1 \text{ / } \delta(t) & 0 \leq t < \Delta \\
0 & \text{ otherwise }
\end{array}
\]

approximates the properties of \(\delta(t)\). By linearity

\[
H(x(t)) = \lim_{\Delta \to 0} \sum_{n} \hat{x}(n \Delta) H(\delta_{\Delta}(t-n \Delta)) \Delta
\]

which evaluated as an integral gives

\[
H(x(t)) = \int_{0}^{T} \hat{x}(\tau) H(\delta(t-\tau)) d \tau
\]

Since \(H(\delta(t-\tau))\) is the shifted unit impulse response \((h(t-\tau))\), this gives the result

\[
H(x(t)) = \int_{0}^{T} \hat{x}(\tau) \hat{h}(t-\tau) d \tau = (x * h)(t).
\]

Hence, circular convolution has been defined such that the output of a linear time invariant system is given by the convolution of the system input with the system unit impulse response.

**Graphical Intuition**

It is often helpful to be able to visualize the computation of a circular convolution in terms of graphical processes. Consider the circular convolution of two finite length functions \(f\), \(g\) given by

\[
(f * g)(t) = \int_{0}^{T} \hat{f}(\tau) \hat{g}(t-\tau) d \tau = \int_{0}^{T} \hat{f}(t-\tau) \hat{g}(\tau) d \tau
\]

The first step in graphically understanding the operation of convolution is to plot each of the periodic extensions of the functions. Next, one of the functions must be selected, and its plot reflected across the \(\tau=0\) axis. For each \(t \in \mathbb{R}[0, T]\), that same function must be shifted left by \(\tau\). The product of the two resulting plots is then constructed. Finally, the area under the resulting curve on \(\mathbb{R}[0, T]\) is computed.

**Convolution Demonstration**

![Convolution Demonstration](https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Signal_Processing_and_Modeling/Book%3A_Signals_and_Syst...)

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Convolution Summary

Convolution, one of the most important concepts in electrical engineering, can be used to determine the output signal of a linear time invariant system for a given input signal with knowledge of the system’s unit impulse response. The operation of continuous time convolution is defined such that it performs this function for infinite length continuous time signals and systems. The operation of continuous time circular convolution is defined such that it performs this function for finite length and periodic continuous time signals. In each case, the output of the system is the convolution or circular convolution of the input signal with the unit impulse response.