6.2: Continuous Time Fourier Series (CTFS)

Introduction

In this module, we will derive an expansion for continuous-time, periodic functions, and in doing so, derive the **Continuous Time Fourier Series (CTFS)**.

Since complex exponentials (Section 1.8) are eigenfunctions of linear time-invariant (LTI) systems (Section 14.5), calculating the output of an LTI system \( \mathscr{H} \) given \( e^{st} \) as an input amounts to simple multiplication, where \( H(s) \in \mathbb{C} \) is the eigenvalue corresponding to \( s \). As shown in the figure, a simple exponential input would yield the output

\[
y(t) = H(s) e^{st}
\]

![Simple LTI system](https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Signal_Processing_and_Modeling/Book%3A_Signals_and_Systems%2F...)

Using this and the fact that \( \mathscr{H} \) is linear, calculating \( y(t) \) for combinations of complex exponentials is also straightforward.

\[
\begin{align*}
c_{1} e^{s_{1} t} + c_{2} e^{s_{2} t} & \rightarrow c_{1} H(s_{1}) e^{s_{1} t} + c_{2} H(s_{2}) e^{s_{2} t} \\
\sum_{n} c_{n} e^{s_{n} t} & \rightarrow \sum_{n} c_{n} H(s_{n}) e^{s_{n} t}
\end{align*}
\]
The action of \(\mathbf{\mathcal{H}}\) on an input such as those in the two equations above is easy to explain. \(\mathbf{\mathcal{H}}\) independently scales each exponential component \(e^{s_{n}t}\) by a different complex number \(H(s_{n}) \in \mathbb{C}\). As such, if we can write a function \(f(t)\) as a combination of complex exponentials it allows us to easily calculate the output of a system.

**Fourier Series Synthesis**

Joseph Fourier demonstrated that an arbitrary \(f(t)\) can be written as a linear combination of harmonic complex sinusoids

\[
f(t) = \sum_{n=-\infty}^{\infty} c_{n} e^{j \omega_{0} n t} \tag{6.3}
\]

where \(\omega_{0} = \frac{2 \pi}{T}\) is the fundamental frequency. For almost all \(f(t)\) of practical interest, there exists \(c_{n}\) to make Equation \ref{6.3} true. If \(f(t)\) is finite energy \(f(t) \in L^{2}[0, T]\), then the equality in Equation \ref{6.3} holds in the sense of energy convergence; if \(f(t)\) is continuous, then Equation \ref{6.3} holds pointwise. Also, if \(f(t)\) meets some mild conditions (the Dirichlet conditions), then Equation \ref{6.3} holds pointwise everywhere except at points of discontinuity.

The \(c_{n}\) - called the Fourier coefficients - tell us "how much" of the sinusoid \(e^{j \omega_{0} n t}\) is in \(f(t)\). The formula shows \(f(t)\) as a sum of complex exponentials, each of which is easily processed by an LTI system (since it is an eigenfunction of every LTI system). Mathematically, it tells us that the set of complex exponentials \(\left\{\forall n, n \in \mathbb{Z}: e^{j \omega_{0} n t}\right\}\) form a basis for the space of \(T\)-periodic continuous time functions.

Example \(\PageIndex{1}\)

We know from Euler's formula that \(\cos(\omega t) + \sin(\omega t) = \frac{1-j}{2} e^{j \omega t} + \frac{1+j}{2} e^{-j \omega t}\).
Synthesis with Sinusoids Demonstration

Figure \(\PageIndex{2}\): Interact (when online) with a Mathematica CDF demonstrating sinusoid synthesis. To download, right click and save as .cdf.

Fourier Series Analysis

Finding the coefficients of the Fourier series expansion involves some algebraic manipulation of the synthesis formula. First of all we will multiply both sides of the equation by \(e^{-\left(j \omega_{0} k t\right)}\), where \(k \in \mathbb{Z}\).

\[
f(t) e^{-\left(j \omega_{0} k t\right)} = \sum_{n=-\infty}^{\infty} c_{n} e^{j \omega_{0} n t} e^{-\left(j \omega_{0} k t\right)}
\]

\(\label{6.4}\)

Now integrate both sides over a given period, \(T\):

\[
\int_{0}^{T} f(t) e^{-\left(j \omega_{0} k t\right)} \, dt = \int_{0}^{T} \sum_{n=-\infty}^{\infty} c_{n} e^{j \omega_{0} n t} e^{-\left(j \omega_{0} k t\right)} \, dt
\]

\(\label{6.5}\)

On the right-hand side we can switch the summation and integral and factor the constant out of the integral.

\[
\int_{0}^{T} f(t) e^{-\left(j \omega_{0} k t\right)} \, dt = \int_{0}^{T} \sum_{n=-\infty}^{\infty} c_{n} e^{j \omega_{0} n t} e^{-\left(j \omega_{0} k t\right)} \, dt
\]

\(\label{6.6}\)

Now that we have made this seemingly more complicated, let us focus on just the integral, \(\int_{0}^{T} e^{j \omega_{0} (n-k) t} \, dt\), on the right-hand side of the above equation. For this integral we will need to consider two cases: \(n=k\) and \(n \neq k\). For \(n=k\) we will have:
For \( n \neq k \), we will have:

\[
\forall n, n \neq k:\left(\int_{0}^{T} e^{j \omega_{0}(n-k) t} \, dt=T\right) \label{6.8}\]

But \( \cos(\omega_0(n−k)t)) \) has an integer number of periods, \( \langle n−k \rangle \), between 0 and \( \langle T \rangle \). Imagine a graph of the cosine; because it has an integer number of periods, there are equal areas above and below the x-axis of the graph. This statement holds true for \( \langle \sin(\omega_0(n−k)t) \rangle \) as well. What this means is

\[
\int_{0}^{T} \cos \left(\omega_{0}(n-k) t\right) \, dt=0\]

which also holds for the integral involving the sine function. Therefore, we conclude the following about our integral of interest:

\[
\int_{0}^{T} e^{j \omega_{0}(n-k) t} \, dt=\begin{cases} 
T & \text{if } n=k \\
0 & \text{otherwise} 
\end{cases}\]

Example \( \PageIndex{2} \)

Consider the square wave function given by

\[
x(t)=\begin{cases} 
1/2 & t \leq 1/2 \\
-1/2 & t>1/2 
\end{cases} \]
Thus, the Fourier coefficients of this function found using the Fourier series analysis formula are

\[
\begin{array}{cc}
\text{k odd} & -j / \pi k \\
\text{k even} & 0
\end{array}
\]

**Fourier Series Summary**

Because complex exponentials are eigenfunctions of LTI systems, it is often useful to represent signals using a set of complex exponentials as a basis. The continuous time Fourier series synthesis formula expresses a continuous time, periodic function as the sum of continuous time, discrete frequency complex exponentials.

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \omega_0 n t}
\]

The continuous time Fourier series analysis formula gives the coefficients of the Fourier series expansion.

\[
c_n = \frac{1}{T} \int_{0}^{T} f(t) e^{-j \omega_0 n t} dt
\]

In both of these equations \(\omega_0 = \frac{2 \pi}{T}\) is the fundamental frequency.