12.7: Rational Functions and the Z-Transform

Introduction

When dealing with operations on polynomials, the term **rational function** is a simple way to describe a particular relationship between two polynomials.

Definition: Rational Function

For any two polynomials, A and B, their quotient is called a rational function.

Example \(\PageIndex{1}\)

Below is a simple example of a basic rational function, \(f(x)\). Note that the numerator and denominator can be polynomials of any order, but the rational function is undefined when the denominator equals zero.

\[f(x)=\frac{x^2-4}{2x^2+x-3} \label{12.48}\]

If you have begun to study the Z-transform, you should have noticed by now they are all rational functions. Below we will look at some of the properties of rational functions and how they can be used to reveal important characteristics about a z-transform, and thus a signal or LTI system.

Properties of Rational Functions

In order to see what makes rational functions special, let us look at some of their basic properties and characteristics. If you are familiar with rational functions and basic algebraic properties, skip to the next subsection to see how rational
functions are useful when dealing with the z-transform.

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**Roots**

To understand many of the following characteristics of a rational function, one must begin by finding the roots of the rational function. In order to do this, let us factor both of the polynomials so that the roots can be easily determined. Like all polynomials, the roots will provide us with information on many key properties. The function below shows the results of factoring the above rational function, Equation \ref{12.48}.

\[
f(x)=\frac{(x+2)(x-2)}{(2 x+3)(x-1)} \quad \text{\ref{12.49}}
\]

Thus, the roots of the rational function are as follows:

Roots of the numerator are: \{-2,2\}

Roots of the denominator are: \{-3,1\}

**Note**

In order to understand rational functions, it is essential to know and understand the roots that make up the rational function.

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**Discontinuities**

Because we are dealing with division of two polynomials, we must be aware of the values of the variable that will cause the denominator of our fraction to be zero. When this happens, the rational function becomes undefined, i.e. we have a discontinuity in the function. Because we have already solved for our roots, it is very easy to see when this occurs. When the variable in the denominator equals any of the roots of the denominator, the function becomes undefined.

Example \ref{PageIndex2}

Continuing to look at our rational function above, Equation \ref{12.48}, we can see that the function will have discontinuities at the following points: \{-3,1\}

In respect to the Cartesian plane, we say that the discontinuities are the values along the x-axis where the function is undefined. These discontinuities often appear as **vertical asymptotes** on the graph to represent the values where the function is undefined.

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**Domain**

Using the roots that we found above, the **domain** of the rational function can be easily defined.

**Definition: Domain**

The group, or set, of values that are defined by a given function.
Example 12.48

Using the rational function above, Equation \ref{12.48}, the domain can be defined as any real number \(x\) where \(x\) does not equal 1 or negative 3. Written out mathematically, we get the following:

\[\{x \in \mathbb{R} \mid (x \neq -3) \text{ and } (x \neq 1)\} \label{12.50}\]

Intercepts

The **x-intercept** is defined as the point(s) where \(f(x)\), i.e. the output of the rational functions, equals zero. Because we have already found the roots of the equation this process is very simple. From algebra, we know that the output will be zero whenever the numerator of the rational function is equal to zero. Therefore, the function will have an x-intercept wherever \(x\) equals one of the roots of the numerator.

The **y-intercept** occurs whenever \(x\) equals zero. This can be found by setting all the values of \(x\) equal to zero and solving the rational function.

Rational Functions and the Z-Transform

As we have stated above, all z-transforms can be written as rational functions, which have become the most common way of representing the z-transform. Because of this, we can use the properties above, especially those of the roots, in order to reveal certain characteristics about the signal or LTI system described by the z-transform.

Below is the general form of the z-transform written as a rational function:

\[X(z)=\frac{b_0+b_1 z^{-1}+\cdots+b_M z^{-M}}{a_0+a_1 z^{-1}+\cdots+a_N z^{-N}} \label{12.51}\]

If you have already looked at the module about Understanding Pole/Zero Plots and the Z-transform (Section 12.5), you should see how the roots of the rational function play an important role in understanding the z-transform. The equation above, Equation \ref{12.51}, can be expressed in factored form just as was done for the simple rational function above, see Equation \ref{12.49}. Thus, we can easily find the roots of the numerator and denominator of the z-transform. The following two relationships become apparent:

Relationship of Roots to Poles and Zeros

- The roots of the numerator in the rational function will be the **zeros** of the z-transform
- The roots of the denominator in the rational function will be the **poles** of the z-transform

Conclusion

Once we have used our knowledge of rational functions to find its roots, we can manipulate a z-transform in a number of useful ways. We can apply this knowledge to representing an LTI system graphically through a Pole/Zero Plot (Section 12.5), or to analyze and design a digital filter through Filter Design from the Z-Transform (Section 12.9).