12.8: Difference Equations

Introduction

One of the most important concepts of DSP is to be able to properly represent the input/output relationship to a given LTI system. A linear constant-coefficient difference equation (LCCDE) serves as a way to express just this relationship in a discrete-time system. Writing the sequence of inputs and outputs, which represent the characteristics of the LTI system, as a difference equation help in understanding and manipulating a system.

Definition: Difference Equation

An equation that shows the relationship between consecutive values of a sequence and the differences among them. They are often rearranged as a recursive formula so that a systems output can be computed from the input signal and past outputs.

Example

\[y[n]+7y[n-1]+2y[n-2]=x[n]-4x[n-1]\]

General Formulas for the Difference Equation

As stated briefly in the definition above, a difference equation is a very useful tool in describing and calculating the output of the system described by the formula for a given sample \(y[n]\). The key property of the difference equation is its ability to help easily find the transform, \(H(z)\), of a system. In the following two subsections, we will look at the general form of the difference equation and the general conversion to a z-transform directly from the difference equation.
Difference Equation

The general form of a linear, constant-coefficient difference equation (LCCDE), is shown below:

\[
\sum_{k=0}^{N} a_{k} y[n-k]=\sum_{k=0}^{M} b_{k} x[n-k] \label{12.52}
\]

We can also write the general form to easily express a recursive output, which looks like this:

\[
\dot{y}[n]=-\sum_{k=1}^{N} a_{k} y[n-k]+\sum_{k=0}^{M} b_{k} x[n-k] \label{12.53}
\]

From this equation, note that \(\dot{y}[n-k]\) represents the outputs and \(\dot{x}[n-k]\) represents the inputs. The value of \(N\) represents the order of the difference equation and corresponds to the memory of the system being represented. Because this equation relies on past values of the output, in order to compute a numerical solution, certain past outputs, referred to as the initial conditions, must be known.

Conversion to Z-Transform

Using the above formula, Equation \ref{12.53}, we can easily generalize the transfer function, \(H(z)\), for any difference equation. Below are the steps taken to convert any difference equation into its transfer function, i.e. z-transform. The first step involves taking the Fourier Transform of all the terms in Equation \ref{12.53}. Then we use the linearity property to pull the transform inside the summation and the time-shifting property of the z-transform to change the time-shifting terms to exponentials. Once this is done, we arrive at the following equation: \(a_0=1\).

\[
\begin{align}
Y(z) &= -\sum_{k=1}^{N} a_{k} Y(z) z^{-k} + \sum_{k=0}^{M} b_{k} X(z) z^{-k} \\
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{\sum_{k=0}^{M} b_{k} z^{-k}}{1+\sum_{k=1}^{N} a_{k} z^{-k}}
\end{align}
\]

Conversion to Frequency Response

Once the z-transform has been calculated from the difference equation, we can go one step further to define the frequency response of the system, or filter, that is being represented by the difference equation.

Note

Remember that the reason we are dealing with these formulas is to be able to aid us in filter design. A LCCDE is one of the easiest ways to represent FIR filters. By being able to find the frequency response, we will be able to look at the basic properties of any filter represented by a simple LCCDE.

Below is the general formula for the frequency response of a z-transform. The conversion is simple a matter of taking the z-transform formula, \(H(z)\), and replacing every instance of \(z\) with \(e^{j\omega}\).

\[
\begin{align}
\end{align}
\]
\[
H(w) &= \left. H(z) \right|_{z= e^{jw}} \\
&= \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}
\]

Once you understand the derivation of this formula, look at the module concerning Filter Design from the Z-Transform (Section 12.9) for a look into how all of these ideas of the Z-transform, Difference Equation, and Pole/Zero Plots (Section 12.5) play a role in filter design.

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**Example**

Example \(\PageIndex{2}\): Finding Difference Equation

Below is a basic example showing the opposite of the steps above: given a transfer function one can easily calculate the systems difference equation.

\[
H(z) = \frac{(z+1)^2}{(z-\frac{1}{2})(z+\frac{3}{4})}
\]

Given this transfer function of a time-domain filter, we want to find the difference equation. To begin with, expand both polynomials and divide them by the highest order \(\left|z\right|\).

\[
\begin{align}
H(z) &= \frac{(z+1)(z+1)}{(z-\frac{1}{2})(z+\frac{3}{4})} \\
&= \frac{z^2+2z+1}{z^2+2z+1-\frac{3}{8}} \\
&= \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}
\end{align}
\]

From this transfer function, the coefficients of the two polynomials will be our \((a_k)\) and \((b_k)\) values found in the general difference equation formula, Equation \ref{12.53}. Using these coefficients and the above form of the transfer function, we can easily write the difference equation:

\[
x[n]+2x[n-1]+x[n-2]=y[n]+\frac{1}{4}y[n-1]-\frac{3}{8}y[n-2]
\]

In our final step, we can rewrite the difference equation in its more common form showing the recursive nature of the system.

\[
y[n]=x[n]+2x[n-1]+x[n-2]+\frac{-1}{4}y[n-1]+\frac{3}{8}y[n-2]
\]

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**Solving a LCCDE**

In order for a linear constant-coefficient difference equation to be useful in analyzing a LTI system, we must be able to find the systems output based upon a known input, \((x(n))\), and a set of initial conditions. Two common methods exist for solving a LCCDE: the **direct method** and the **indirect method**, the later being based on the z-transform. Below we will briefly discuss the formulas for solving a LCCDE using each of these methods.
Direct Method

The final solution to the output based on the direct method is the sum of two parts, expressed in the following equation:

\[ y(n) = y_h(n) + y_p(n) \]

The first part, \( y_h(n) \), is referred to as the **homogeneous solution** and the second part, \( y_p(n) \), is referred to as **particular solution**. The following method is very similar to that used to solve many differential equations, so if you have taken a differential calculus course or used differential equations before then this should seem very familiar.

Homogeneous Solution

We begin by assuming that the input is zero, \( x(n) = 0 \). Now we simply need to solve the homogeneous difference equation:

\[ \sum_{k=0}^{N} a_k y[n-k] = 0 \]

In order to solve this, we will make the assumption that the solution is in the form of an exponential. We will use lambda, \( \lambda \), to represent our exponential terms. We now have to solve the following equation:

\[ \sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \]

We can expand this equation out and factor out all of the lambda terms. This will give us a large polynomial in parenthesis, which is referred to as the **characteristic polynomial**. The roots of this polynomial will be the key to solving the homogeneous equation. If there are all distinct roots, then the general solution to the equation will be as follows:

\[ y_h(n) = C_1 \left( \lambda_1 \right)^n + C_2 \left( \lambda_2 \right)^n + \ldots + C_N \left( \lambda_N \right)^n \]

However, if the characteristic equation contains multiple roots then the above general solution will be slightly different. Below we have the modified version for an equation where \( \lambda_1 \) has \( K \) multiple roots:

\[ y_h(n) = C_1 \left( \lambda_1 \right)^n + C_1 n \left( \lambda_1 \right)^n + C_1 n^2 \left( \lambda_1 \right)^n + \ldots + C_1 n^{K-1} \left( \lambda_1 \right)^n + C_2 \left( \lambda_2 \right)^n + \ldots + C_N \left( \lambda_N \right)^n \]

Particular Solution

The particular solution, \( y_p(n) \), will be any solution that will solve the general difference equation:

\[ \sum_{k=0}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \]

In order to solve, our guess for the solution to \( y_p(n) \) will take on the form of the input, \( x(n) \). After guessing at a solution to the above equation involving the particular solution, one only needs to plug the solution into the difference equation and solve it out.
Indirect Method

The indirect method utilizes the relationship between the difference equation and z-transform, discussed above, to find a solution. The basic idea is to convert the difference equation into a z-transform, as described above, to get the resulting output, \(Y(z)\). Then by inverse transforming this and using partial-fraction expansion, we can arrive at the solution.

\[Z\{y(n+1)-y(n)\}=z (Y(z)-y(0))\]

This can be iteratively extended to an arbitrary order derivative as in Equation \ref{12.69}.

\[Z\left\{-\sum_{m=0}^{N-1} y(n-m)\right\}=z^{N} Y(z)-\sum_{m=0}^{N-1} z^{N-m-1} y^{(m)}(0) \tag{12.69}\]

Now, the Laplace transform of each side of the differential equation can be taken

\[Z\left\{\sum_{k=0}^{N} a_{k}\left[y(n-m+1)-\sum_{m=0}^{N-1} y(n-m) y(n)\right]\right\}=Z\{x(n)\}\]

which by linearity results in

\[\sum_{k=0}^{N} a_{k} Z\left\{y(n-m+1)-\sum_{m=0}^{N-1} y(n-m) y(n)\right\}=Z\{x(n)\}\]

and by differentiation properties in

\[\sum_{k=0}^{N} a_{k} \left(z^{k} Z\{y(n)\}-\sum_{m=0}^{N-1} z^{k-m-1} y^{(m)}(0)\right)=Z\{x(n)\}.\]

Rearranging terms to isolate the Laplace transform of the output,

\[Y(z)=\frac{X(z)+\sum_{k=0}^{N} \sum_{m=0}^{k-1} a_{k} z^{k-m-1} y^{(m)}(0)}{\sum_{k=0}^{N} a_{k} z^{k}}. \tag{12.74}\]

In order to find the output, it only remains to find the Laplace transform \(X(z)\) of the input, substitute the initial conditions, and compute the inverse Z-transform of the result. Partial fraction expansions are often required for this last step. This may sound daunting while looking at Equation \ref{12.74}, but it is often easy in practice, especially for low order difference equations. Equation \ref{12.74} can also be used to determine the transfer function and frequency response.

As an example, consider the difference equation

\[y[n-2]+4 y[n-1]+3 y[n]=\cos (n)\]

with the initial conditions \(y(0)=1\) and \(y(0)=0\) Using the method described above, the Z transform of the solution \(y[n]\) is given by

\[Y[z]=\frac{1}{z^{2}+1}(1+\cos [z+3]+\frac{1}{z+3})\]
Performing a partial fraction decomposition, this also equals
\[Y[z]=.25 \frac{1}{z+1}-.35 \frac{1}{z+3}+.1 \frac{z}{z^2+1}+.2 \frac{1}{z^2+1}.\]

Computing the inverse Laplace transform,
\[y(n)=\left(.25 z^{-n}-.35 z^{-3 n}+.1 \cos (n)+.2 \sin (n)\right) u(n).\]

One can check that this satisfies both the differential equation and the initial conditions.