1.4: An Electro-Mechanical System Model

Model of a DC Motor

A electro-mechanical system converts electrical energy into mechanical energy or vice versa. A armature-controlled DC motor (Figure 1.4.1) represents such a system, where the input is the armature voltage, \( V_a(t) \), and the output is motor speed, \( \omega(t) \), or angular position \( \theta(t) \).

In order to develop a model of the DC motor, let \( i_a(t) \) denote the armature current, and \( L \) and \( R \) denote the electrical side inductance and the coil resistance. The mechanical side inertia and friction are denoted as \( J \) and \( b \), respectively. Let \( k_t \) denote the torque constant and \( k_b \) the motor constant; then, the dynamic equations of the DC motor are given as:

\[
L \frac{d i_a(t)}{dt} + R i_a(t) + k_b \omega(t) = V_a(t) \\
J \frac{d \omega(t)}{dt} + b \omega(t) - k_t i_a(t) = 0
\]

By using the Laplace transform, these equations are transformed into algebraic equations as:

\[
(Ls + R) i_a(s) + k_b \omega(s) = V_a(s) \\
J \omega(s) + b \omega(s) - k_t i_a(s) = 0
\]

\[
(sL + R) i_a(s) + k_b \omega(s) = V_a(s) \\
(sJ + b) \omega(s) - k_t i_a(s) = 0
\]
Motor Transfer Function

In order to obtain an input-output relation for the DC motor, we may solve the first equation for \(i_a(s)\) and substitute in the second equation. Alternatively, we multiply the first equation by \(k_t\), the second equation by \((Ls+R)\), and add them together to obtain:

\[
(Ls+R)(Js+b)\omega(s) + k_t k_b \omega(s) = k_t V_a(s)
\]

Then, the transfer function of the DC motor with voltage input and angular velocity output is derived as:

\[
\frac{\omega(s)}{V_a(s)} = \frac{k_t}{(Ls+R)(Js+b)+k_t k_b}
\]

The denominator polynomial in the DC motor transfer function typically has real roots, which are reciprocals of the motor time constants \((\tau_e, \tau_m)\). In terms of the time constants, the DC motor model is described as:

\[
\frac{\omega(s)}{V_a(s)} = \frac{k_t / JL}{(s+1/\tau_e)(s+1/\tau_m)}
\]

The electrical constant represents the build up of electrical current in the armature circuit, whereas the mechanical constant represents the build-up of motor speed in response to the developed motor torque. Further, the slower mechanical time constant dominates the overall motor response to a change in the armature voltage.

The angular position \(\theta(s)\) of the shaft is obtained by integrating the angular velocity \(\omega(s)\); the transfer function from \(V_a(s)\) to the angular displacement \(\theta(s)\) is given as:

\[
\frac{\theta(s)}{V_a(s)} = \frac{k_t}{s[(Ls+R)(Js+b)+k_t k_b]}
\]

Example

A small DC motor has the following parameter values: \(R=1\Omega, L=0.01H, J=0.01\ kgm^2, b=0.1\ \frac{N-s}{rad}, k_t = k_b = 0.05\); then, the motor transfer function from armature voltage to angular velocity is obtained as:

\[
\frac{\omega(s)}{V_a(s)} = \frac{500}{(s+100)(s+10)+25} = \frac{500}{(s+10.28)(s+99.72)}
\]

The two motor time constants are given as: \(\tau_e \cong 10\ ms, \tau_m \cong 100\ ms\), where \(\tau_e\) matches the time constant of an RL circuit \(\tau_e = L/R\) and \(\tau_m\) matches the time constant of inertial mass in the presence of friction \(\tau_m = J/b\).

Assuming a unit-step input, \(u(s) = \frac{1}{s}\), is applied to the motor, the motor speed is obtained as:
\[ \omega(s) = \frac{500}{s(s+10.28)(s+99.72)} = \frac{0.488}{s} - \frac{0.544}{s+10.28} + \frac{0.056}{s+99.72} \]

By applying the inverse Laplace transform, the time-domain output is given as (Figure 13a):

\[ \omega(t) = \left[0.488 - 0.544e^{-10.28t} + 0.056e^{-99.72t}\right]u(t) \]

where \(u(t)\) denotes a unit-step function. The motor response is plotted in Figure 1.4.2.

**Simplified Model of a DC motor**

A simplified model of the DC motor is obtained by ignoring the coil inductance \((L \to 0)\). Then, the electrical side equation is modified as:

\[ Ri_a(s) + k_b \omega(s) = V_a(s) \]

By substituting \(i_a(s)\) into the torque equation, the mechanical side equation is given as:

\[ R(Js+b)\omega(s) + k_t k_b \omega(s) = k_t V_a(s) \]

The resulting first-order motor transfer function is given as:

\[ \frac{\omega(s)}{V_a(s)} = \frac{k_t/R}{Js+b+k_t k_b/R} \]

The first-order model has a single motor time constant \(\tau_m = \frac{JR}{bR+k_t k_b}\), and is written as:

\[ \frac{\omega(s)}{V_a(s)} = \frac{k_t/R}{s+1/\tau_m} \]

**Example 1.4.2**

Using the parameter values for a small DC motor (Example 1.4.1), its reduced first-order transfer function is obtained as:

\[ \frac{\omega(s)}{V_a(s)} = \frac{5}{s+10.25} \]

The resulting motor time constant evaluates as: \(\tau_m \approx 97.6\; ms\), which approximates the slower mechanical time constant in the second-order model.

Assuming a unit-step input, the motor response is obtained as:

\[ \omega(s) = \frac{500}{s(s+10.25)(s+99.72)} = \frac{0.488}{s} - \frac{0.488}{s+10.25} \]

By applying the inverse Laplace transform, the motor output is given as:

\[ \omega(t) = \left[0.488 - 0.488e^{-10.25t}\right]u(t) \]

The motor response to a unit-step input is plotted in Figure 1.4.2.
Figure \(\PageIndex{2}\): DC motor response to unit-step input: second-order motor model (left); first-order motor model (right).

```plaintext
pkg load control
R=1; L=.01; J=.01; b=.1; kt=.05; kb=.05;
s=tf('s');
Gs=kt/((L*s+R)*(J*s+b)+kt*kb)
step(Gs), hold
L=0;
Gs1=kt/((L*s+R)*(J*s+b)+kt*kb)
step(Gs1)
```