2.4: The Step Response

Step Response

Definition: Step Response

The step response of a system is defined as its response to a unit-step input, \( u(t), u(s) = \frac{1}{s} \).

Let \( G(s) \) describe the system transfer function; then, the unit-step response is obtained as: \( y(s) = G(s) \frac{1}{s} \). The inverse Laplace transform leads to: \( y(t) = \mathcal{L}^{-1}\left[ \frac{G(s)}{s} \right] \).

Alternatively, the step response can be obtained by integrating the impulse response: \( y(t) = \int_0^t g(t - \tau) d\tau \).

The unit-step response of a system starts from an initial value: \( y(0) = y_0 \), and settles at a steady-state value: \( y(\infty) = \lim_{t \to \infty} y(t) \). Further, from the application of the final value theorem (FVT): \( y(\infty) = G(0) \).

The unit step response in the case of the first- and second-order systems is described below.

**First-Order System.** Let \( G(s) = \frac{K}{\tau s + 1}, u(s) = \frac{1}{s} \); then, \( y(s) = \frac{K}{s} - \frac{K\tau}{\tau s + 1} \).

The time-domain response is given as: \( y(t) = K(1 - e^{-t/\tau}) \).

Assuming arbitrary initial conditions, \( y(0) = y_0 \), the step response is given as:
Example (PageIndex{1})

Let \(G(s)=\frac{1}{2s+1}\); then, the unit-step response is obtained as: \(y(s)=\frac{1}{s(2s+1)}\). The time-domain response is given as: \(y(t)=(1-e^{-t/2} )\, u(t)\).

First-Order System with an Integrator. Let \(G(s)=\frac{K}{s(\tau s+1)}\); then, \(y(s)=\frac{K}{s^2(\tau s+1)}\). Using PFE, we obtain: \(y(t)=K(t-\tau (1-e^{-t/\tau} ))\, u(t)\). The step response grows out of bound as \(t\to \infty\).

Example (PageIndex{2})

Let \(G(s)=\frac{1}{s(2s+1)}\); then, the unit-step response is computed as: \(y(s)=\frac{1}{s^2(2s+1)}\). The time-domain response is given as: \(y(t)=(t-2+2e^{-t/2} )\, u(t)\).
**Second-Order System with Real Poles.** Let \( G(s)=\frac{K}{(\tau _1 s+1)(\tau _2 s+1)} \), with \( \tau _1 > \tau _2 \); then, the step response is computed as: \( y(s)=\frac{A}{s} + \frac{B}{\tau _1 s+1} + \frac{C}{\tau _2 s+1} \). Hence, \[ y(t)=(A+Be^{-t/\tau _1 } +Ce^{-t/\tau _2 } )u(t) \]

where \( A=K,B=-\frac{K}{\tau _1 -\tau _2},C=\frac{K}{\tau _2 -\tau _1} \).

**Example \( \PageIndex{3} \)**

A small DC motor has the following parameter values: \( R=1\; \Omega ,\; L=10\; {mH},\; J=0.01\; {kg-m}^{2},\; b=0.1\; \frac{N-s}{rad},\; k_{t}=k_{b}=0.05\). The motor transfer function, from armature voltage to motor speed, is approximated as: \( G(s)=\frac{500}{(s+10)(s+100)} \). The step response of the motor is obtained as: \( y(s)=\frac{1}{2s}-\frac{0.556}{s+10}+\frac{0.056}{s+100} \). The time-domain response is given as: \( y(t)=(0.5-0.556e^{-10t}+0.056e^{-100t})u(t) \), which settles at \( y_{\infty }=0.5 \). The natural response modes, \( e^{-10t} \) and \( e^{-100t} \), reflect motor electrical and mechanical time constants: \( \tau _e \approx 0.01s \) and \( \tau _m \approx 0.1s \). The slower mechanical time constant dominates the motor step response.

![Step Response](https://eng.libretexts.org/Bookshelves/Industrial_and_Systems_Engineering/Book%3A_Introduction_to_Control_Systems_(Iq…

**Second-Order System with Complex Poles.** Let \( G(s)=\frac{K}{(s+\sigma )^{2} +\omega _d^{2} } \). Then, the unit-step response is computed as: \( y(s)=\frac{A}{s} + \frac{Bs+C}{(s+\sigma )^{2} +\omega _d^{2} } \), where \( A=G(0)=\frac{K}{\sigma ^{2} +\omega _d^{2} },B=-A,C=-2A\sigma \). Hence,

\[ y(t)=\frac{K}{\sigma ^{2} +\omega _d^{2} } \left[1-e^{-\sigma t} \left(\cos \; \omega _d t+\frac{\sigma }{\omega _d } \sin \; \omega _d t\right)\right]u(t) \]

In the phase form, the unit-step response is given as:

\[ y(t)=\frac{K}{\sigma ^{2} +\omega _d^{2} } \left[1-e^{-\sigma t} \cos \; \left(\omega _d t-\phi \right)\right]u(t) \]

where \( \phi =\tan ^{-1} \frac{\sigma }{\omega _d } \).
Example \(\PageIndex{4}\))

A mass–spring–damper system has the following parameter values: \(m=1\ \text{kg},\ b=6\frac{\text{Ns}}{\text{m}},\ k=25\frac{\text{N}}{\text{m}}\). Its transfer function is given as: \(G(s)=\frac{1}{(s+3)^2+4^2}\). The system has complex poles located at: \(s=-3\pm j4\).

The unit-step response of the system is computed as: \(y(s)=\frac{1}{25}\left[\frac{1}{s}-\frac{s+6}{(s+3)^2+4^2}\right]\). By applying the inverse Laplace transform, the time-domain response is obtained as:

\[
y(t)=\frac{1}{25}\left[1-e^{-3t}\left(\cos(4t)+\frac{3}{4}\sin(4t)\right)\right]u(t)
\]

Figure \(\PageIndex{4}\): Step response of mass–spring–damper model with complex poles.

**System with Dead-time.** The first-order-plus-dead-time (FOPDT) model of an industrial process is given as: \(G(s)=\frac{Ke^{-t_ds}}{\tau s+1}\), where \(\{K,\tau,t_d\}\) represent the process gain, time constant, and dead-time.

The step response of the FOPDT model is computed as: \(y(s)=\frac{Ke^{-t_ds}}{s(\tau s+1)}\), which translated into a time-domain response as: \(y(t)=(1-e^{-(t-t_d)/\tau})u(t-t_d)\).

An approximate process model, obtained by using Pade’ approximation, is given as: \(G(s)=\frac{K(1-t_ds/2)}{(1+t_ds/2)(\tau s+1)}\). This model is termed as non-minimum phase due to the additional phase contributed by the right half-plane (RHP) zero.

The unit-step response of the approximate model is computed as: \(y(s)=\frac{K(1-t_ds/2)}{s(1+t_ds/2)(\tau s+1)}\).

Example \(\PageIndex{5}\))

The FOPDT model of a stirred-tank bio-reactor is given as: \(G(s)=\frac{20e^{-s}}{0.5s+1}\).

The step response of the system is computed as: \(y(s)=\frac{20e^{-s}}{s(0.5s+1)}\). The time-domain response is obtained as: \(y(t)=20\left(1-e^{-2(t-1)}\right)u(t-1)\).

Using a first-order Pade’ approximation, an approximate process model is obtained as: \(G_a(s)=\frac{20(1-0.5s)}{(1+0.5s)(0.5s+1)}\).
right)}{(0.5s+1)^2}.

The step response of the approximate model is computed as: 
\( y(s) = \frac{20(1-0.5s)}{s(0.5s+1)^2} \), 
\( y(t) = 20(1-(1-4t)e^{-2t})u(t) \).

The two responses are compared below (Figure 2.4.5). The step response for the FOPDT model starts after the designated delay. The step response for the Pade' approximation starts with an undershoot due to the presence of RHP zero.

![Step Response Graph](https://eng.libretexts.org/Bookshelves/Industrial_and_Systems_Engineering/Book%3A_Introduction_to_Control_Systems_(Iq...)

Figure \(\PageIndex{5}\): Step response of an industrial process model with dead-time.

We make the following observations based on the figure:

1. The step response of the process with dead-time starts after 1 s delay (as expected).
2. The step response of Pade' approximation of delay has an undershoot. This behavior is characteristic of transfer function models with zeros located in the right-half plane.