6.1: Frequency Response Plots

Frequency Response Function

The frequency response of the loop transfer function, \(KGH(s)\), is represented as: \(KGH(j\omega) = |KGH(j\omega)|e^{j\phi(\omega)}\). For a particular value of \(\omega\), the \(KGH(j\omega)\) is a complex number, which may be described in terms of its magnitude and phase as \(KGH(j\omega) = |KGH(j\omega)|e^{j\phi(\omega)}\).

As \(\omega\) varies from \(0\) to \(\infty\), \(KGH(j\omega)\) can be plotted in the complex plane (the polar plot). Alternatively, both magnitude and phase can be plotted as functions of \(\omega\) (the Bode magnitude and phase plots).

To proceed further, we assume that the loop transfer function is expressed as:

\[
KGH(s) = \frac{K \prod_{i=1}^{m} \left(1+\frac{s}{z_i} \right)}{s^{n_0} \prod_{i=1}^{n_1} \left(1+\frac{s}{p_i} \right) \prod_{i=1}^{n_2} \left(1+2\zeta_i \frac{s}{\omega_{n,i}} + \frac{s^2}{\omega_{n,i}^2} \right)}
\]

where \(m\) is the number of real zeros, \(n_0\) is the number of poles at the origin, \(n_1\) is the number of real poles, \(n_2\) is the number of complex pole pairs, and \(K\) is a scalar gain. Then,

\[
KGH(j\omega) = \frac{K \prod_{i=1}^{m} \left(1+j\frac{\omega}{z_i} \right)}{(j\omega)^{n_0} \prod_{i=1}^{n_1} \left(1+j\frac{\omega}{p_i} \right) \prod_{i=1}^{n_2} \left(1-\frac{\omega^2}{\omega_{n,i}^2} + j2\zeta_i \frac{\omega}{\omega_{n,i}} \right)}
\]

Bode Plot

It is customary to plot the Bode magnitude plot on log scale as \(|G(j\omega)|_{\text{dB}} = 20\log_{10}|G(j\omega)|\).
Thus, for the frequency response function:

\[
\begin{array}{r}
|KGH(j\omega)|_{\text{dB}} = 20 \log K + \sum_{i=1}^{m} 20 \log \left|1+\frac{j\omega}{z_{i}}\right| - (20n_{0})\log \omega \\
- \sum_{i=1}^{n_{1}} 20 \log \left|1+\frac{j\omega}{p_{i}}\right| - \sum_{i=1}^{n_{2}} 20 \log \left|1-\frac{\omega^{2}}{\omega_{n,i}^{2}} + j2\zeta_{i}\frac{\omega}{\omega_{n,i}}\right|
\end{array}
\]

At low frequencies, the magnitude plot has: \(|KGH(j\omega)|_{\text{dB}} = 20 \log K\). For large \(|\omega|\), the magnitude plot is characterized by a slope: \((-20(n-m))\text{dB/decade}\), where \(n-m\) represents the pole excess of the loop transfer function.

The phase angle \(|\phi(\omega)|\) of the loop transfer function is computed as:

\[
|\phi(\omega)| = \sum_{i=1}^{m} \angle \left(1+\frac{j\omega}{z_{i}}\right) - n_{0} (90^\circ) - \sum_{i=1}^{n_{1}} \angle \left(1+\frac{j\omega}{p_{i}}\right) - \sum_{i=1}^{n_{1}} \angle \left(1-\frac{\omega^{2}}{\omega_{n,i}^{2}} + j2\zeta_{i}\frac{\omega}{\omega_{n,i}}\right).
\]

The phase angle for large \(|\omega|\) is given as: \(|\phi(\omega)| = -90^\circ (n-m)\).

Bode magnitude and phase plots for higher order transfer functions are composition of Bode plots of first and second order factors, which facilitated their hand plotting in the past.

In the MATLAB Control Systems Toolbox, the Bode plot is obtained using the “bode” command, invoked after defining the transfer function.

Nyquist Plot

The frequency response function \(|KGH(j\omega)|\) represents a function of a complex variable. A polar plot describes the graph of \(|KGH(j\omega)|\) \(\omega\) varies from \(0\to \infty\). The Nyquist plot is a closed curve that describes a graph of \(|KGH(j\omega)|\) for \(|\omega| \in \left(-\infty, \infty\right)\).

The shape of the Nyquist plot depends on the poles and zeros of \(|KGH(j\omega)|\); it can be pictured by estimating magnitude and phase of \(|KGH(j\omega)|\) at low and high frequencies. For example, if \(|KGH(s)|\) has no poles at the origin, then at low frequency, \(|KGH(j0)|\cong K|\angle 0^\circ|\); while, at high frequency, \(|KGH(j\infty)|\cong |K|\angle -90^\circ (n-m)|\), where \(n-m\) represents the pole excess of \(|KGH(s)|\).

In particular, for \(|n-m=3|\), the Nyquist plot crosses the negative real-axis at the phase crossover frequency, \(|\omega|_{pc}\). Let \(|G|\left|\text{arg}(\omega_{pc})=90^\circ\right|\); then, the gain margin is given as \(|g|\left|\text{arg}(\omega_{pc})=90^\circ\right|\). Further, for \(|K=g|\left|\text{arg}(\omega_{pc})=90^\circ\right|\), the Nyquist plot of \(|G|\left|\text{arg}(\omega_{pc})=90^\circ\right|\) passes through the \((-1+j0)\), described as the critical point for stability determination.

Definition: Nyquist Stability Criterion

The number of unstable closed-loop poles of \(|\Delta(s)|=1+KGH(s)|\)equals the number of unstable open-loop poles of \(|KGH(s)|\) plus the number of clock-wise (CW) encirclements of the \((-1+j0)\) point on the complex plane by the Nyquist plot of \(|KGH(s)|\).
In the MATLAB Control Systems Toolbox the Nyquist plot is obtained by invoking the ‘nyquist’ command, invoked after defining the transfer function:

Example \( \PageIndex{1} \)

Let \( G(s) = \frac{1}{s+1} \); then, \( G(j\omega) = \frac{1}{\sqrt{1+\omega^2}} \ \angle -\tan^{-1} \omega \).

In particular, \( G(0) = 1; \ \angle 0^\circ \), \( G(1) = \frac{1}{\sqrt{2}} \angle -45^\circ \), and \( G(\infty) = 0 \ \angle -90^\circ \).

The Bode magnitude plot (Figure 6.1) starts at \( 0 \ dB \) with an initial slope of zero that gradually changes to \( -20 \ dB \) per decade at high frequencies. The phase plot varies from \( 0^\circ \) to \( -90^\circ \) with a phase of \( -45^\circ \) at the corner frequency.

The Nyquist plot of \( G(s) \) is a circle in the right-half plane (RHP). Further, the Nyquist plot of \( KG(j\omega) \) is confined to the RHP for positive values of \( K \); hence the closed-loop system is stable for all \( K > 0 \).

Figure 6.1: Bode and Nyquist plots for \( G(s) = \frac{1}{s+1} \)

Example \( \PageIndex{2} \)

Let \( G(s) = \frac{1}{s(s+1)} \); then, \( G(j\omega) = \frac{1}{\omega} \frac{1}{\sqrt{1+\omega^2}} \ \angle -90^\circ -\tan^{-1} \omega \).

In particular, \( G(0) = \infty \angle -90^\circ \), \( G(1) = \frac{1}{\sqrt{2}} \angle -135^\circ \), \( G(\infty) = 0 \angle -180^\circ \).

The Bode magnitude plot (Figure 6.2) has an initial slope of \( -20 \ dB \) per decade that gradually changes to \( -40 \ dB \) per decade at high frequencies. The phase plot shows a variation from \( -90^\circ \) to \( -180^\circ \) with a phase of \( -135^\circ \) at the corner frequency.

The Nyquist plot is a closed curve that travels along negative \( j\omega \)-axis for \( \omega \in (0,\infty) \), along positive \( j\omega \)-axis for \( \omega \in (-\infty,0) \), and scribes a semi-circle of a large radius for \( \omega \in (0^-,0^+) \). As the Nyquist plot of \( KG(j\omega) \) stays away from the critical point \((-1+j0)\) for positive values of \( K \), the closed-loop system is projected to be stable for all \( K > 0 \).

Figure 6.2: Bode and Nyquist plots for \( G(s) = \frac{1}{s(s+1)} \)

Example \( \PageIndex{3} \)

Let \( G(s) = \frac{1}{s^2+s+1} \); then, \( G(j\omega) = \frac{1}{\sqrt{(1-\omega^2)^2+\omega^2}} \ \angle -\tan^{-1} \frac{\omega}{1-\omega^2} \).

In particular, \( G(0) = 1 \angle 0^\circ \), \( G(1) = 1 \angle -90^\circ \), \( G(\infty) = 0 \angle -180^\circ \).

Example \( \PageIndex{4} \)

Let \( G(s) = \frac{1}{s^2+2s+3} \); then, \( G(j\omega) = \frac{1}{\sqrt{(1-\omega^2)^2+2\omega^2}} \ \angle -\tan^{-1} \frac{\omega}{1-\omega^2} \).

In particular, \( G(0) = 1 \angle 0^\circ \), \( G(1) = 1 \angle -90^\circ \), \( G(\infty) = 0 \angle -180^\circ \).

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The Bode magnitude plot (Figure 6.3) starts at \(0\ dB\) with an initial slope of 0; the slope changes to \((-40\ dB)\) per decade for \(\omega > 1\).

The Nyquist plot of \((G(j\omega))\) is a closed curve that has no crossing with the negative real-axis. As the Nyquist plot of \((KG(j\omega)), K>0\) stays away from the critical point \((-1+j0))\), the closed-loop system is projected to be stable for all \(K>0\).

**Figure 6.3: Bode and Nyquist plots for \((G(s))=\frac{1}{s^2+s+1}\).**

**Example 4**

Let \(G(s)=\frac{2}{s(s+1)(s+2)}\); then, \(G(j\omega )=\frac{2}{j\omega (1+j\omega)(2+j\omega)} =\frac{1}{\omega} \frac{1}{\sqrt{1+\omega^2}} \frac{2}{\sqrt{2+\omega^2}} \angle -90^\circ -\tan^{-1}\omega -\tan^{-1}2\omega\).

In particular, \(G(0)=\infty \angle -90^\circ\), \(G(1)=\frac{2}{\sqrt{10}} (-4dB) \angle -108^\circ\), \(G(j1)=\frac{1}{5\sqrt{2}} (-13dB) \angle -139^\circ\), and \(G(\infty)=0dB \angle -270^\circ\).

The Bode magnitude plot (Figure 6.4) has an initial slope of \((-20/\text{dB})\) per decade that changes first to \((-40/\text{dB})\) per decade and then to \((-60/\text{dB/decade})\) at high frequencies. The phase plot shows a variation from \(-90^\circ\) to \(-270^\circ\).

The polar plot begins with a large magnitude along the negative \(j\omega\)-axis (for \(\omega = 0^+\)), crosses the negative real-axis at \(0.33\angle 180^\circ\) (for \(\omega = 3.32\)), and approaches the origin from the positive \(j\omega\)-axis (for \(\omega = \infty\))(Figure 6.3). The Nyquist plot further includes a reflection of the polar plot for \(\omega \in (-\infty,0)\).

The Nyquist plot includes a closed contour located in the left-half plane that includes a real-axis crossing at \(0.33\angle 180^\circ\), the closed-loop system is stable for \(K<3\).

**Figure 6.4: Bode and Nyquist plots for \((G(s))=\frac{2}{s(s+1)(s+2)}\).**