7.6: Digital Controller Design by Emulation

7.6.1 Emulation of Analog Controller

The controller for a sampled-data control system can be designed by emulating (i.e., approximating) an available analog controller design. Controller emulation aims to obtain an approximate digital controller, \( \left( K \left( z \right) \right) \), whose response matches that of the analog controller, \( \left( K \left( s \right) \right) \). Popular controller emulation methods include:

**Impulse Invariance.** The impulse invariance method aims to match the impulse response of the analog controller. Assume that the analog controller has a form: \( H(s) = \sum_{k=1}^{n} \frac{A_{k}}{s-s_{k}} \). Then a digital controller is obtained as: \( H(z) = T \sum_{k=1}^{n} \frac{A_{k}}{1-p_{k}z^{-1}} \), \( p_{k} = e^{s_{k}T} \).

**Pole–Zero Matching.** In pole–zero matching, both the plant poles and zeros are mapped to their equivalent locations in the \( z \)-plane using: \( z_{i} = e^{s_{i}T} \). If the analog transfer function has \( n \) poles and \( m \) finite zeros, to be matched, then another \( n-m \) zeros are added at \( z=1 \). Additionally, the dc gain of the digital controller is matched to that of the analog controller.

**Zero-Order Hold (ZOH).** The ZOH method assumes the presence of a ZOH at the input. The method is ineffective if the analog controller has poles at the origin (as in the case of PI and PID controllers).

**First-Order Hold (FOH).** The FOH method assumes a piece-wise linear input to the controller.

**Bilinear Transform (BLT).** The BLT method uses Tustin’s approximation \( \left( \frac{1+Ts/2}{1-Ts/2} \right) \) to emulate an analog controller. The BLT method is effective with a high sampling frequency. Using the BLT, an equivalent digital controller is obtained as: \( K(z) = K(s) \left| s = \frac{2}{T} \frac{z-1}{z+1} \right. \).

The ‘c2d’ command in the MATLAB Control Systems Toolbox allows controller emulation method to be specified. The

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default method is ZOH.

Example \(\PageIndex{1}\)

Let \(G(s) = \frac{10}{s(s+2)(s+5)}\); \(T = 0.2s\); then, the pulse transfer function is obtained as: 
\[
G(z) = \frac{0.0095(z+0.18)(z+2.68)}{(z-1)(z-0.67)(z-0.37)}
\]

A lead-lag controller for this example was designed earlier: 
\[
K(s) = 25\left(\frac{s+2}{s+24}\right)\left(\frac{s+0.05}{s+0.004}\right)
\]

We use MATLAB Control System Toolbox 'c2d' commands to obtain discrete approximations of the controller using the following methods:

**ZOH:** 
\[
K(z) = \frac{25(z-0.99)(z-0.925)}{(z-0.999)(z-0.008)}
\]

**FOH:** 
\[
K(z) = \frac{6.86(z-0.99)(z-0.7)}{(z-0.999)(z-0.008)}
\]

**BLT:** 
\[
K(z) = \frac{8.86(z-0.99)(z-0.667)}{(z-0.999)(z-0.412)}
\]

**Impulse-invariance:** 
\[
K(z) = \frac{-109.77z(z-0.999)}{(z-0.999)(z-0.008)}
\]

**Pole-zero matching:** 
\[
K(z) = \frac{6.3(z-0.99)(z-0.67)}{(z-0.999)(z-0.008)}
\]

**Least-squares:** 
\[
K(z) = \frac{5.74(z-0.99)(z-0.542)}{(z-0.999)(z-0.376)}
\]

An analysis of the closed-loop systems shows that all except the impulse invariance approximation are stable with damping in the range of \(\zeta \in [0.72, 0.81]\). The closed-loop responses are compared in Figure 7.8.

As seen from the figure and otherwise, the BLT method generally provides the best approximation of an analog controller.

Figure 7.8: Step response comparison for analog controller emulation methods (Example 7.22).

### 7.6.2 Emulation of Analog PID Controller

An analog PID controller with input \(e(t)\) and output \(u(t)\) is described as:

\[
u(t) = k_p e(t) + k_d \frac{de(t)}{dt} + k_i \int e(t) dt.
\]

An equivalent digital PID controller can be obtained by employing a forward Euler approximation of the derivative term:

\[
\frac{de(t)}{dt} \approx \frac{1}{T}(e_{k+1} - e_k).
\]

The resulting \(z\)-domain transfer functions for the PID controller are given as:

\[
K(z) = k_p + k_d(z-1) + k_i \frac{T}{z-1}
\]

Further, let \(v_{-1}(k)\) denote the integrator output; then, the digital PID controller is implemented using the following
update rules:
\[ u_k = k_{pe_{k-1}} + \frac{1}{T}(e_k - e_{k-1}) + k_{iv_k} \]
\[ v_k = v_{k-1} + Te_{k-1} \]

Example \( \PageIndex{2} \)

Let \( G(s) = \frac{10}{s(s+2)(s+5)}, \ T = 0.2s \); then, the pulse transfer function is obtained as:
\[ G(z) = \frac{0.0095(z+0.18)(z+2.68)}{(z-1)(z-0.67)(z-0.37)} \]

A PID controller for this example was designed earlier as:
\[ K_{PID}(s) = \frac{1.2(s+0.05)(s+2)}{s} \]

An approximate digital controller is obtained as:
\[ K_{PID}(z) = 2.46 + 1.2 \left( \frac{z-1}{T} \right) + 0.12 \left( \frac{T}{z-1} \right) \]

The step response of the closed-loop system for analog and digital PID controllers is shown in Figure 7.9. The discrete PID controller with forward Euler approximation of derivative emulates the analog controller well.

Figure 7.9: Comparison of the step response for analog and digital PID controllers.