2.5: Transformations between Inertial Frames

In extending empirically determined conduction, polarization and magnetization laws to include material motion, it is often necessary to relate field variables evaluated in different reference frames. A given point in space can be designated either in terms of the coordinate \(\overrightarrow{r}\) or of the coordinate \(\overrightarrow{r'}\) of Figure 2.5.1. By "inertial reference frames," it is meant that the relative velocity between these two frames is constant, designated by \(\overrightarrow{u}\). The positions in the two coordinate systems are related by the Galilean transformation:

\[
\overrightarrow{r'} = \overrightarrow{r} - \overrightarrow{u}t; \quad t' = t
\]

Figure 2.5.1 Reference frames have constant relative velocity \(\overrightarrow{u}\). The coordinates \(\overrightarrow{r} = (x,y,z)\) and \(\overrightarrow{r'} = (x',y',z')\) designate the same position.
It is a familiar fact that variables describing a given physical situation in one reference frame will not be the same as those in the other. An example is material velocity, which, if measured in one frame, will differ from that in the other frame by the relative velocity $\overrightarrow{u}$.

There are two objectives in this section: one is to show that the quasistatic laws are invariant when subject to a Galilean transformation between inertial reference frames. But, of more use is the relationship between electromagnetic variables in the two frames of reference that follows from this proof. The approach is as follows. First, the postulate is made that the quasistatic equations take the same form in the primed and unprimed inertial reference frames. But, in writing the laws in the primed frame, the spatial and temporal derivatives must be taken with respect to the coordinates of that reference frame, and the dependent field variables are then fields defined in that reference frame. In general, these must be designated by primes, since their relation to the variables in the unprimed frame is not known.

For the the purpose of writing the primed equations of electrodynamics in terms of the unprimed coordinates, recognize that

$$\begin{align}
\nabla^{'} &\rightarrow \nabla \\
\frac{\partial{\overrightarrow{A}}}{\partial{t^{'}}} &\rightarrow (\frac{\partial}{\partial{t}} + \overrightarrow{u} \cdot \nabla) \overrightarrow{A} \equiv \frac{\partial{\overrightarrow{A}}}{\partial{t}} + \overrightarrow{u} \nabla \cdot \overrightarrow{A} - \nabla \times (\overrightarrow{u} \times \overrightarrow{A}) \\
\frac{\partial{\psi}}{\partial{t^{'}}} &\rightarrow (\frac{\partial}{\partial{t}} + \overrightarrow{u} \cdot \nabla) \psi \equiv \frac{\partial{\psi}}{\partial{t}} + \nabla \cdot \overrightarrow{u} \psi
\end{align}$$

The left relations follow by using the chain rule of differentiation and the transformation of Equation 1. That the spatial derivatives taken with respect to one frame must be the same as those with respect to the other frame physically means that a single "snapshot" of the physical process would be all required to evaluate the spatial derivatives in either frame. There would be no way of telling which frame was the one from which the snapshot was taken. By contrast, the time rate of change for an observer in the primed frame is, by definition, taken with the primed spatial coordinates held fixed. In terms of the fixed frame coordinates, this is the convective derivative defined with Eqs. 2.4.5 and 2.4.6. However, $(\overrightarrow{\nabla} \psi)$ in these equations is in general a function of space and time. In the context of this section it is specialized to the constant $(\overrightarrow{\nabla} \psi)$. Thus, in rewriting the convective derivatives of Equation 2 the constancy of $(\overrightarrow{\nabla} \psi)$ and a vector identity (Equation 16, Appendix B) have been used.

So far, what has been said in this section is a matter of coordinates. Now, a physically motivated postulate is made concerning the electromagnetic laws. Imagine one electromagnetic experiment that is to be described from the two different reference frames. The postulate is that provided each of these frames is inertial, the governing laws must take the same form. Thus, Eqs. 2.4.23-27 apply with $(\Delta \rightarrow \Delta^{'}, \partial() / \partial{t} \rightarrow \partial() / \partial{t^{'}})$ and all dependent variables primed. By way of comparing these laws to those expressed in the fixed-frame, Eqs. 2.4.2 are used to rewrite these expressions in terms of the unprimed independent variables. Also, the moving-frame material velocity is rewritten in terms of the unprimed frame velocity using the relation

$$\overrightarrow{v^{'}} = \overrightarrow{v} - \overrightarrow{u} \quad \text{(Equation 3)}$$

Thus, the laws originally expressed in the primed frame of reference become

$$\nabla \cdot \varepsilon_o \overrightarrow{E^{'}} = -\nabla \cdot \overrightarrow{P^{'}} + \rho_f^{'} \quad \text{(Equation 4)}$$
\[ \nabla \times \overrightarrow{H^{'}} = \overrightarrow{J_f^{'}} \quad \label{4} \]
\[ \nabla \times \overrightarrow{E^{'}} = 0 \quad \quad \nabla \cdot \mu_o \overrightarrow{H^{'}} = -\nabla \cdot \mu_o \overrightarrow{M^{'}} \quad \label{5} \]
\[ \nabla \cdot (\overrightarrow{J_f^{'}} + \overrightarrow{u} \rho_f^{'}) + \frac{\partial{\rho_f^{'}}}{\partial{t}} = 0 \quad \quad \nabla \times (\overrightarrow{E^{'}} - \overrightarrow{u} \times \mu_o \overrightarrow{H^{'}}) = -\frac{\partial{\mu_o \overrightarrow{H^{'}}}}{\partial{t}} -\frac{\partial{\mu_o \overrightarrow{M^{'}}}}{\partial{t}} -\mu_o \nabla \times (\overrightarrow{M^{'}} \times \overrightarrow{v}) \quad \label{6} \]
\[ \nabla \times (\overrightarrow{H^{'}} + \overrightarrow{u} \times \varepsilon_o \overrightarrow{E^{'}}) = (\overrightarrow{J_f^{'}} + \overrightarrow{u} \rho_f^{'}) + \frac{\partial{\varepsilon_o \overrightarrow{E^{'}}}}{\partial{t}} + \frac{\partial{\overrightarrow{P^{'}}}}{\partial{t}} + \nabla \times (\overrightarrow{P^{'}} \times \overrightarrow{v}) \quad \quad \nabla \cdot \overrightarrow{J_f^{'}} = 0 \quad \label{7} \]
\[ \nabla \cdot \mu_o \overrightarrow{H^{'}} = -\nabla \mu_o \overrightarrow{M^{'}} \quad \quad \nabla \cdot \varepsilon_o \overrightarrow{E^{'}} = -\nabla \cdot \overrightarrow{P^{'}} + \rho_f^{'} \quad \label{8} \]

In writing Equation \ref{7}a, Equation \ref{4}a is used. Similarly, Equation \ref{5}b is used to write Equation \ref{6}b. For the one experiment under consideration, these equations will predict the same behavior as the fixed frame laws, Eqs. 2.3.23-27, if the identification is made:
\[ \overrightarrow{E^{'}} = \overrightarrow{E} \quad \quad \overrightarrow{H^{'}} = \overrightarrow{H} \quad \label{9} \]
\[ \overrightarrow{P^{'}} = \overrightarrow{P} \quad \quad \overrightarrow{M^{'}} = \overrightarrow{M} \quad \label{10} \]
\[ \rho_f^{'} = \rho_f \quad \quad \overrightarrow{J_f^{'}} = \overrightarrow{J_f} \quad \label{11} \]
\[ \overrightarrow{J_f^{'}} = \overrightarrow{J_f} - \overrightarrow{u} \rho_f \quad \quad \overrightarrow{E^{'}} = \overrightarrow{E} + \overrightarrow{u} \times \mu_o \overrightarrow{H} \quad \label{12} \]
\[ \overrightarrow{H^{'}} = \overrightarrow{H} - \overrightarrow{u} \times \varepsilon_o \overrightarrow{E} \quad \quad \overrightarrow{D^{'}} = \overrightarrow{D} \quad \quad \overrightarrow{B^{'}} = \overrightarrow{B} \quad \label{14} \]

The primary fields are the same whether viewed from one frame or the other. Thus, the EQS electric field polarization density and charge density are the same in both frames, as are the MQS magnetic field, magnetization density and current density. The respective dynamic laws can be associated with those field transformations that involve the relative velocity. That the free current density is altered by the relative motion of the net free charge in the EQS system is not surprising. But, it is the contribution of this same convection current to Ampere's law that generates the velocity dependent contribution to the EQS magnetic field measured in the moving frame of reference. Similarly, the velocity dependent contribution to the MQS electric field transformation is a direct consequence of Faraday's law.

The transformations, like the quasistatic laws from which they originate, are approximate. It would require Lorentz transformations to carry out a similar procedure for the exact electrodynamical laws of Sec. 2.2. The general laws are not invariant in form to a Galilean transformation, and therein is the origin of special relativity. Built in from the start in the quasistatic field laws is a self-consistency with other Galilean invariant laws describing mechanical continua that will be brought in in later chapters.