6.28: Inclined Pipes

In dredging inclined pipes occur in ladders of cutter suction dredgers and suction pipes of trailing suction hopper dredgers. On land inclined pipes occur going up and down slopes. So inclined pipes may have positive and negative inclination angles up to 45°. The question is, what is the influence of the inclination angle on the hydraulic gradient, on the Limit of Stationary Deposit Velocity (LSDV) and on the Limit Deposit Velocity (LDV). The effect of inclined pipes is expressed based on the length of the pipe, not the horizontal distance. A number of cases have to be distinguished.

6.28.1 Pure Carrier Liquid

First of all, the flow of pure carrier liquid. The equilibrium of forces on the liquid is:

\[-\frac{dp}{dx} \cdot A \cdot L = \tau_l \cdot O \cdot L + \rho_l \cdot A \cdot L \cdot g \cdot \sin(\theta)\]

The hydraulic gradient can now be determined with:

\[-\frac{dp}{dx} \cdot \frac{A \cdot L}{\rho_l \cdot A \cdot L \cdot g} = \frac{\tau_l \cdot O \cdot L}{\rho_l \cdot A \cdot L \cdot g} + \frac{\rho_l \cdot A \cdot L \cdot g \cdot \sin(\theta)}{\rho_l \cdot A \cdot L \cdot g} = i_l + \sin(\theta)\]

So apparently the hydraulic gradient increases with the sine of the inclination angle. Which also means that a downwards slope with a negative inclination angle gives a negative sine and thus a reduction of the hydraulic gradient. In this case the hydraulic gradient may even become negative.
6.28.2 Stationary Bed Regime

The equilibrium of forces on the layer of liquid above the bed is:
\[
-\frac{\mathrm{dp}}{\mathrm{dx}} \cdot \mathrm{A}_{1} \cdot \mathrm{L} = \tau_{1} \cdot \mathrm{O}_{1} \cdot \mathrm{L} + \tau_{12} \cdot \mathrm{O}_{12} \cdot \mathrm{L} + \rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g \cdot \sin (\theta)
\]
Since the bed is not moving, the friction between the bed and the pipe wall compensates for the weight component of the bed. The hydraulic gradient can now be determined with:
\[
\mathrm{i}_{\text{m, } \theta} = -\frac{\frac{\mathrm{dp}}{\mathrm{dx}} \cdot \mathrm{A}_{1} \cdot \mathrm{L}}{\rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g} = \frac{\tau_{1} \cdot \mathrm{O}_{1} \cdot \mathrm{L} + \tau_{12} \cdot \mathrm{O}_{12} \cdot \mathrm{L}}{\rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g} + \frac{\rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g \cdot \sin (\theta)}{\rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g} = \mathrm{i}_{\text{m}} + \sin (\theta)
\]
Which is the hydraulic gradient of a stationary bed in a horizontal pipe plus the sine of the inclination angle. The weight of the solids do not give a contribution to the hydraulic gradient, since it is carried by the pipe wall.

6.28.3 Sliding Bed Regime

The equilibrium of forces on the layer of liquid above the bed is:
\[
-\frac{\mathrm{dp}}{\mathrm{dx}} \cdot \mathrm{A}_{1} \cdot \mathrm{L} = \tau_{1} \cdot \mathrm{O}_{1} \cdot \mathrm{L} + \tau_{12} \cdot \mathrm{O}_{12} \cdot \mathrm{L} + \rho_{l} \cdot \mathrm{A}_{1} \cdot \mathrm{L} \cdot g \cdot \sin (\theta)
\]
The cross-section of the bed and the layer of liquid above the bed can be determined with:
\[
\frac{\mathrm{A}_{2}}{\mathrm{A}} = \frac{C_{v_{b}}}{C_{v_{s}}} \quad \text{and} \quad \frac{\mathrm{A}_{1}}{\mathrm{A}} = \frac{\mathrm{A} - \mathrm{A}_{2}}{\mathrm{A}}
\]
The weight of the bed, including pore water is:
\[
\begin{array}{l}
\text{W}_{\text{b}} &= \rho_{b} \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g = \left(\rho_{s} \cdot C_{v_{b}} + \rho_{l} \cdot \left(1 - C_{v_{b}}\right)\right) \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g = \left(\left(\rho_{s} - \rho_{l}\right) \cdot C_{v_{b}} + \rho_{l}\right) \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g = \rho_{l} \cdot R_{sd} \cdot C_{v_{s}} \cdot \mathrm{A} \cdot \mathrm{L} \cdot g + \rho_{l} \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g
\end{array}
\]
The submerged weight of the bed can be determined with:
\[
\text{W}_{\text{s, submerged}} = \rho_{b} \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g = \rho_{l} \cdot R_{sd} \cdot C_{v_{s}} \cdot \mathrm{A} \cdot \mathrm{L} \cdot g + \rho_{l} \cdot \mathrm{A}_{2} \cdot \mathrm{L} \cdot g
\]
This gives for the equilibrium of forces on the bed:

\[
\begin{array}{l}
-\frac{d p}{d x} \cdot A_2 \cdot L = \tau_2 \cdot O_2 \cdot L - \tau_{12} \cdot O_{12} \cdot L + W_{b,s} \cdot \sin \theta + \mu_{sf} \cdot W_{b,s} \cdot \cos \theta
\end{array}
\]

For the whole pipe cross section, the two contributions can be added, giving:

\[
\begin{array}{l}
-\frac{d p}{d x} \cdot A \cdot L = \tau_1 \cdot O_1 \cdot L + \tau_2 \cdot O_2 \cdot L + \rho_{l} \cdot A \cdot L \cdot g \cdot \sin \theta + \mu_{sf} \cdot \rho_{l} \cdot R_{sd} \cdot C_{vs} \cdot A \cdot L \cdot g \cdot \cos \theta
\end{array}
\]

This can also be written as:

\[
\begin{array}{l}
-\frac{d p}{d x} \cdot A \cdot L = \tau_1 \cdot O_1 \cdot L + \tau_2 \cdot O_2 \cdot L + \rho_{m} \cdot A \cdot L \cdot g \cdot \sin \theta + \mu_{sf} \cdot \rho_{l} \cdot R_{sd} \cdot C_{vs} \cdot A \cdot L \cdot g \cdot \cos \theta
\end{array}
\]

In terms of the hydraulic gradient this gives:

\[
\begin{array}{l}
i_{m,\theta} = -\frac{d p}{d x} \cdot \frac{A \cdot L}{\rho_{l} \cdot g \cdot L \cdot A} = \frac{\tau_1 \cdot O_1 \cdot L}{\rho_{l} \cdot g \cdot L \cdot A} + \left(1 + R_{sd} \cdot C_{vs}\right) \cdot \sin \theta + \mu_{sf} \cdot R_{sd} \cdot C_{vs} \cdot \cos \theta
\end{array}
\]

In chapter 7.4 it will be proven that the first term on the right hand side almost equals the pure liquid hydraulic gradient \(i_l\), without pipe inclination, so:

\[
\begin{array}{l}
i_{m,\theta} = i_l + \left(1 + R_{sd} \cdot C_{vs}\right) \cdot \sin \theta + \mu_{sf} \cdot R_{sd} \cdot C_{vs} \cdot \cos \theta
\end{array}
\]
\[ \mathcal{C}_{\text{vs}} \cdot \sin (\theta) + \mu_{\text{sf}} \cdot \mathcal{R}_{\text{sd}} \cdot \mathcal{C}_{\text{vs}} \cdot \cos (\theta) \]

\[ i_{\theta} = (i_l + \sin (\theta)) + \mathcal{R}_{\text{sd}} \cdot \mathcal{C}_{\text{vs}} \cdot (\mu_{\text{sf}} \cdot \cos (\theta) + \sin (\theta)) \]

Giving for the mixture hydraulic gradient with pipe inclination:

\[ i_{\theta} = i_{l, \theta} + \mathcal{R}_{\text{sd}} \cdot \mathcal{C}_{\text{vs}} \cdot (\mu_{\text{sf}} \cdot \cos (\theta) + \sin (\theta)) \]

### 6.28.4 Homogeneous Regime

In the homogeneous flow regime, the hydraulic gradient is:

\[ i_{\theta} = i_{l, \theta} + \mathcal{R}_{\text{sd}} \cdot \mathcal{C}_{\text{vs}} \]

For an inclined pipe only the lifting of the mixture has to be added, giving:

\[ \begin{array}{l}
\left( i_{l} + \sin (\theta) \right) \cdot \left( 1 + \mathcal{R}_{\text{sd}} \cdot \mathcal{C}_{\text{vs}} \right) \\
\text{or} \\
\begin{array}{l}
\mathcal{C}_{\text{vs}} \cdot \left( \mu_{\text{sf}} \cdot \cos (\theta) + \sin (\theta) \right) \\
\end{array}
\end{array} \]

Some researchers found that the hydraulic gradient does not increase for 100% with the solids effect, but just with about 60% of the solids effect. In this case the factor \( A \) should be taken 0.6 instead of 1.0. The solids effect in the inclination part of the equation always counts for 100% however, since it’s the increase or decrease of the potential energy of the mixture.

### 6.28.5 Conclusions So Far

For pure liquid, the fixed bed regime, the sliding bed regime and the homogeneous flow regime, the influence of the inclination angle can be determined fundamentally, based on the spatial volumetric concentration. The latter is very important. For a sliding bed, the force to move the bed upwards depends on the submerged weight of the bed and the inclination angle, but not on the velocity of the bed. For homogeneous flow it is assumed that spatial and transport concentration are almost equal. The question is now, what is the influence of the inclination angle in the heterogeneous flow regime? Different researchers have different methods. These methods are described in the next chapters.
6.28.6 The Heterogeneous Flow Regime, Durand & Condolios and Gibert

The basic equation of Durand & Condolios (1952) and Gibert (1960) for the solids effect is given by:

\[
\Phi = \frac{i_m - i_l}{i_l \cdot C_v t} = 81 \cdot \left(\frac{v_{ls}^2 \cdot \sqrt{C_x}}{g \cdot D_p \cdot R_{sd}}\right)^{-3/2}
\]

For inclined pipes they modified the solids effect by adding the cosine of the inclination angle according to:

\[
\Phi_{\theta} = \frac{i_{m,\theta} - \sin(\theta) \cdot \left(1 + R_{sd} \cdot C_{vt}\right) - i_l}{i_l \cdot C_v t} = 81 \cdot \left(\frac{v_{ls}^2 \cdot \sqrt{C_x}}{g \cdot D_p \cdot R_{sd} \cdot \cos(\theta)}\right)^{-3/2}
\]

This can be written as:

\[
i_{m,\theta} = i_l + \sin(\theta) \cdot \left(1 + R_{sd} \cdot C_{vt}\right) + i_l \cdot 81 \cdot \left(\frac{v_{ls}^2 \cdot \sqrt{C_x}}{g \cdot D_p \cdot R_{sd} \cdot \cos(\theta)}\right)^{-3/2} \cdot C_v t
\]

So the solids effect has to be multiplied with the cosine of the inclination angle to the power of 3/2. This means the solids effect is decreasing with an increasing inclination angle, whether the inclination is upwards or downwards. It should be mentioned that the hydraulic gradient is based on the length of the pipe and not on the horizontal length component.
6.28.7 The Heterogeneous Flow Regime, Worster & Denny

Worster & Denny (1955) have a slightly different approach. They state that the hydraulic gradient in an inclined pipe equals the sum of the hydraulic gradients of the horizontal component and the vertical component. This gives the following equation:

\[
i_{m, \theta} = i_{l, \theta} + i_l \cdot 81 \cdot \left(\frac{v_{ls}^2 \cdot \sqrt{C_x}}{g \cdot D_p \cdot R_{sd}}\right)^{-3/2} \cdot C_{vt} \cdot \cos(\theta) + \sin(\theta) \cdot R_{sd} \cdot C_{vt}
\]

The difference with Durand & Condolios (1952) and Gibert (1960) is the power of the cosine. In both cases, the equations match the hydraulic gradient of a horizontal pipe if the inclination angle equals zero and a vertical pipe if the inclination angle equals 90 degrees, whether the inclination is upwards (positive inclination angle) or downwards (negative inclination angle). The homogeneous component for vertical pipes is still missing here.

6.28.8 The Heterogeneous Flow Regime, Wilson et al

Wilson et al. (2006) derived the following equation for heterogeneous transport in horizontal pipes:

\[
i_m = i_l + \frac{\mu_s}{2} \cdot \left(\frac{v_{50}}{v_{ls}}\right)^M \cdot R_{sd} \cdot C_{vt}
\]

For inclined pipes they modified the equation, matching the reasoning of Worster & Denny (1955), but with the use of the power M according to:

\[
i_{m, \theta} = i_{l, \theta} + \frac{\mu_s}{2} \cdot \left(\frac{v_{50}}{v_{ls}}\right)^M \cdot R_{sd} \cdot C_{vt} \cdot \cos(\theta)^M + \sin(\theta) \cdot R_{sd} \cdot C_{vt}
\]

The power M has a value of 1.7 for uniform or narrow graded sands and decreases to 0.25 for very broad graded sands. For narrow graded sands the influence of the inclination angle is similar to the Durand & Condolios (1952) and Gibert (1960) approach with a power of 1.5 versus 1.7 for Wilson et al. (2006). For medium graded sands with a power around 1, the influence is similar to the Worster & Denny (1955) approach.

6.28.9 The Sliding Bed Regime, Doron et al

Doron et al. (1997) investigated the influence of inclined pipes, based on their 2LM and 3LM models. Basically they multiplied the sliding friction with the cosine of the inclination angle and they added the potential energy term, which is proportional with the sine of the inclination angle. They carried out experiments with inclination angles from -7 to +7 degrees. The resulting data however is dominated by the potential energy term, because of the small inclination angles.
Wilson et al. (2006) use a graph with experimental data for the Deposition Limit. Their Deposition Limit however is the Limit of Stationary Deposit Velocity (LSDV) and not the Limit Deposit Velocity. The LSDV is the line speed where a bed starts sliding, while the LDV is defined in this book as the line speed above which there is no stationary or sliding bed. The LDV is thus always higher than the LSDV. The LSDV does not always exist. For smaller particles it is very well possible that there is a direct transition between the stationary bed regime and the heterogeneous flow regime. The graph used by Wilson et al. (2006) shows an increasing LSDV with increasing inclination angle up to an inclination angle of about 30 degrees above which the LSDV is constant or decreasing. The experimental data stop at an inclination angle of 40 degrees. For negative inclination angles, the LSDV decreases with a decreasing inclination angle. The experimental data stop at an angle of -20 degrees.

The research of Doron et al. (1997) gives a similar result. They also investigated the LSDV and not the LDV. The maximum LSDV was found at about 15 degrees inclination angle.

The behavior of the LSDV for inclined pipes can be explained assuming that the LSDV is the intersection point of the stationary bed regime and the sliding bed regime. In the stationary bed regime the hydraulic gradient is:

\[
\mathrm{i}_{\mathrm{m}, \theta} = \lambda_{\mathrm{m}} \cdot \frac{v_{\mathrm{ls}}^{2}}{(1-C_{\mathrm{vr}})^{2} \cdot 2 \cdot g \cdot D_{\mathrm{p}}}
\]

In the sliding bed regime the hydraulic gradient is:

\[
\mathrm{i}_{\mathrm{m}, \theta} = \lambda_{\mathrm{d}} \cdot \frac{v_{\mathrm{ls}}^{2}}{2 \cdot g \cdot D_{\mathrm{p}}} + \sin \theta + R_{\mathrm{sd}} \cdot C_{\mathrm{vs}} \cdot (\mu_{\mathrm{sf}} \cdot \cos \theta + \sin \theta)
\]

The intersection point (line speed) occurs when both hydraulic gradients are equal, so:

\[
\mathrm{i}_{\mathrm{m}} - \mathrm{i}_{\mathrm{l}} = R_{\mathrm{sd}} \cdot C_{\mathrm{vs}} \cdot (\mu_{\mathrm{sf}} \cdot \cos \theta + \sin \theta)
\]

Assuming \(\lambda_{\mathrm{m}}\) equals the hydraulic gradient in the restricted area above the bed with a Darcy Weisbach friction coefficient \(\lambda_{\mathrm{m}}\) based on the flow above the bed, this gives:

\[
\begin{array}{l}
\lambda_{\mathrm{m}} = \frac{v_{\mathrm{ls}, \mathrm{lsdv}}^{2}}{2 \cdot g \cdot R_{\mathrm{sd}} \cdot D_{\mathrm{p}}} = \frac{\left(\mu_{\mathrm{sf}} \cdot \cos \theta + \sin \theta\right) \cdot C_{\mathrm{vs}}}{\left(\frac{\lambda_{\mathrm{m}}}{(1-C_{\mathrm{vr}})^{2}} - \lambda_{\mathrm{d}}\right)}
\end{array}
\]

The Darcy Weisbach friction coefficient \(\lambda_{\mathrm{m}}\) may be dependent on the size of the particles in the bed. The above derivation is indicative, because suspension and the particle slip velocity are not taken into account. Using a sliding...
friction factor of 0.4 does result in a 40% higher LSDV for an inclination angle of 30 degrees, where the Wilson et al. (2006) graph predicts an increase of 25%-30%. The above equation does show a maximum at 66 degrees inclination angle with a 57% increase of the LSDV.

The experiments of Graf & Robinson (1970) resulted in a modified $F_L$ number for the LDV:

$$F_L = \frac{v_{ls,ldv}}{\sqrt{2 \cdot g \cdot D_p \cdot R_{sd}}} \cdot (1-\tan(\theta))$$

Apparently the LDV decreases with increasing inclination angle for an ascending pipe. The inclination angles used were however very small.

### 6.28.11 The U Tube as a Device to Determine the Delivered Volumetric Concentration

#### 6.28.11.1 The Vertical Ascending Pipe

The mass balance of the solids flow gives:

$$A_p \cdot C_{vt} \cdot v_{ls} \cdot \rho_s = A_p \cdot C_{vs} \cdot v_s \cdot \rho_s = A_p \cdot C_{vs} \cdot (v_{l} - v_{th}) \cdot \rho_s$$

Or:

$$C_{vt} \cdot v_{ls} = C_{vs} \cdot v_s = C_{vs} \cdot (v_{l} - v_{th})$$

In this mass balance it is assumed that the particles (solids) have a velocity $v_s$ smaller than the cross sectional averaged line speed $v_{ls}$ and the liquid (water) has a larger velocity $v_l$. The mass balance of the liquid (water) gives:

$$A_p \cdot (1-C_{vt}) \cdot v_{ls} \cdot \rho_l = A_p \cdot (1-C_{vs}) \cdot v_{l} \cdot \rho_l$$

Or:

$$(1-C_{vt}) \cdot v_{ls} = (1-C_{vs}) \cdot v_{l}$$

This gives for the liquid velocity $v_l$, assuming the terminal hindered settling velocity $v_{th}$ is known:

$$v_{l} = v_{ls} + C_{vs} \cdot v_{th}$$

https://eng.libretexts.org/Bookshelves/Civil_Engineering/Book%3A_Slurry_Transport_(Miedema)/06%3A_Slurry_Transport_a… Updated: Tue, 22 Sep 2020 14:31:35 GMT Powered by
If the spatial volumetric concentration $C_{vs}$ is known, the delivered or transport volumetric concentration $C_{vt}$ can be determined according to:

\[
C_{vt} = C_{vs} \cdot \left(1 - \frac{v_{th}}{v_{ls}} \right) + C_{vs}^2 \cdot \frac{v_{th}}{v_{ls}}
\]

If the delivered volumetric concentration $C_{vt}$ is known, the spatial volumetric concentration $C_{vs}$ can be determined according to:

\[
C_{vs} = -\frac{1}{2} \cdot \left(\frac{v_{ls}}{v_{th}} - 1\right) + \frac{1}{2} \cdot \sqrt{\left(\frac{v_{ls}}{v_{th}} - 1\right)^2 + 4 \cdot C_{vt} \cdot \frac{v_{ls}}{v_{th}}}
\]

For low concentrations and/or particles with a small terminal settling velocity this can be approximated by:

\[
C_{vt} = C_{vs} \cdot \left(1 - \frac{v_{th}}{v_{ls}} \right)
\]

And:

\[
C_{vs} = C_{vt} \cdot \left(\frac{v_{ls}}{v_{ls} - v_{th}} \right)
\]

In an ascending vertical pipe the spatial volumetric concentration $C_{vs}$ is larger than the delivered volumetric concentration $C_{vt}$.

Since the liquid velocity is higher than the line speed, depending on the spatial concentration and the terminal hindered settling velocity, the line speed in the hydraulic gradient equation should be replaced by the liquid velocity, assuming that the Darcy Weisbach friction factor hardly changes at high Reynolds numbers. This gives for the hydraulic gradient of a mixture in a vertical pipe:

\[
i_{m} = \frac{\rho_{m}}{\rho_{l}} \cdot \frac{\lambda_{l} \cdot \left(v_{ls} + C_{vs} \cdot v_{th}\right)^2}{2 \cdot g \cdot D_p} = \frac{\rho_{m}}{\rho_{l}} \cdot \frac{\lambda_{l} \cdot \left(v_{ls}^2 + 2 \cdot C_{vs} \cdot v_{th} \cdot v_{ls} + C_{vs}^2 \cdot v_{th}^2\right)}{2 \cdot g \cdot D_p}
\]

\[
i_{m} \approx \frac{\rho_{m}}{\rho_{l}} \cdot \frac{\lambda_{l} \cdot v_{ls}^2}{2 \cdot g \cdot D_p}
\]
So in an ascending vertical pipe the Darcy Weisbach friction losses are larger than the losses found based on the equivalent liquid model.

### 6.28.11.2 The Vertical Descending Pipe

The mass balance of the solids flow gives:

\[
\frac{\rho_{m}}{\rho_{l}} \cdot \frac{2 \cdot \lambda_{l} \cdot C_{v s} \cdot v_{t h} \cdot v_{l}}{2 \cdot g \cdot D_{p}}
\]

So in an ascending vertical pipe the Darcy Weisbach friction losses are larger than the losses found based on the equivalent liquid model.

The mass balance of the solids flow gives:

\[
A_{p} \cdot C_{v t} \cdot v_{l s} \cdot \rho_{s} = A_{p} \cdot C_{v s} \cdot v_{s} \cdot \rho_{s} = A_{p} \cdot C_{v s} \cdot (v_{l} + v_{t h}) \cdot \rho_{s}
\]

Or:

\[
C_{v t} \cdot v_{l s} = C_{v s} \cdot (v_{l} + v_{t h})
\]

In this mass balance it is assumed that the particles (solids) have a velocity \( v_{s} \) smaller than the cross sectional averaged line speed \( v_{ls} \) and the liquid (water) has a larger velocity \( v_{l} \). The mass balance of the liquid (water) gives:

\[
A_{p} \cdot (1 - C_{v t}) \cdot v_{l s} \cdot \rho_{l} = A_{p} \cdot (1 - C_{v s}) \cdot v_{l} \cdot \rho_{l}
\]

Or:

\[
(1 - C_{v t}) \cdot v_{l s} = (1 - C_{v s}) \cdot v_{l}
\]

This gives for the liquid velocity \( v_{l} \), assuming the terminal hindered settling velocity \( v_{th} \) is known:

\[
v_{l} = v_{l s} - C_{v s} \cdot v_{t h}
\]

If the spatial volumetric concentration \( C_{vs} \) is known, the delivered or transport volumetric concentration \( C_{vt} \) can be determined according to:

\[
C_{vt} = C_{vs} \cdot (1 + \frac{v_{t h}}{v_{ls}}) - C_{vs}^2 \cdot \frac{v_{t h}}{v_{ls}}
\]

If the delivered volumetric concentration \( C_{vt} \) is known, the spatial volumetric concentration \( C_{vs} \) can be determined.
according to:

\[
C_{vs} = \frac{1}{2} \left( \frac{v_{ls}}{v_{th}} + 1 \right) + \frac{1}{2} \sqrt{\left( \frac{v_{ls}}{v_{th}} + 1 \right)^2 - 4 \cdot C_{vt} \cdot \frac{v_{ls}}{v_{th}}} \]

For low concentrations and/or particles with a small terminal settling velocity this can be approximated by:

\[
C_{vt} = C_{vs} \cdot \left( 1 + \frac{v_{th}}{v_{ls}} \right)
\]

And:

\[
C_{vs} = C_{vt} \cdot \left( \frac{v_{ls}}{v_{ls} + v_{th}} \right)
\]

In a descending vertical pipe the delivered volumetric concentration \( C_{vt} \) is larger than the spatial volumetric concentration \( C_{vs} \).

Since the liquid velocity is smaller than the line speed, depending on the spatial concentration and the terminal hindered settling velocity, the line speed in the hydraulic gradient equation should be replaced by the liquid velocity, assuming that the Darcy Weisbach friction factor hardly changes at high Reynolds numbers. This gives for the hydraulic gradient of a mixture in a vertical pipe:

\[
i_m = \frac{\rho_m}{\rho_l} \cdot \frac{\lambda_l \cdot \left( v_{ls} - C_{vs} \cdot v_{th} \right)^2}{2 \cdot g \cdot D_p} = \frac{\rho_m}{\rho_l} \cdot \frac{\lambda_l \cdot \left( v_{ls}^2 - 2 \cdot C_{vs} \cdot v_{th} \cdot v_{ls} + C_{vs}^2 \cdot v_{th}^2 \right)}{2 \cdot g \cdot D_p}
\]

So in an ascending vertical pipe the Darcy Weisbach friction losses as larger than the losses found based on the equivalent liquid model.

6.28.11.3 Determination of the Delivered Volumetric Concentration

Now suppose in both the ascending and the descending pipes the hydraulic gradients are measured with differential
pressure transducers. The delivered volumetric concentration $C_{vt}$ is equal in both pipes. Using the index $a$ for the ascending pipe and the index $d$ for the descending pipe, this gives for the ascending pipe:

\[
\begin{align*}
\rho_{m_{a}} &= \frac{\rho_{l}}{(1+R_{sd}C_{vs_{a}})} \\
C_{vs_{a}} &= C_{vt} \left( \frac{v_{ls}}{v_{ls}-v_{th}} \right) \\
\rho_{m_{d}} &= \frac{\rho_{l}}{(1+R_{sd}C_{vs_{d}})} \\
C_{vs_{d}} &= C_{vt} \left( \frac{v_{ls}}{v_{ls}+v_{th}} \right)
\end{align*}
\]

The difference of the two hydraulic gradients is now:

\[
\begin{align*}
\begin{array}{rl}
\rho_{m_{a}} &= \rho_{l} \\
\rho_{m_{d}} &= \rho_{l}
\end{array}
\end{align*}
\]

This gives:

\[
\begin{align*}
\begin{array}{rl}
\rho_{m_{a}} &= \rho_{l} \\
\rho_{m_{d}} &= \rho_{l}
\end{array}
\end{align*}
\]
\[ i_{m,a} - i_{m,d} = \left( \frac{\rho_{m,a}}{\rho_l} - \frac{\rho_{m,d}}{\rho_l} \right) \cdot \frac{\lambda_1 \cdot v_{ls}^2}{2 \cdot g \cdot D_p} + \left( \frac{\rho_{m,a}}{\rho_l} \cdot \frac{2 \cdot \lambda_l \cdot C_{v,t} \cdot v_{th} \cdot v_{ls}}{2 \cdot g \cdot D_p} + \frac{\rho_{m,d}}{\rho_l} \cdot \frac{2 \cdot \lambda_l \cdot C_{v,d} \cdot v_{th} \cdot v_{ls}}{2 \cdot g \cdot D_p} \right) + \left( \frac{\rho_{m,a}}{\rho_l} + \frac{\rho_{m,d}}{\rho_l} \right) \]

If the terminal hindered settling velocity is very small compared to the line speed, giving an average mixture density based on the delivered concentration, this can be simplified to:

\[ i_{m,a} - i_{m,d} = 2 \cdot \frac{\rho_m}{\rho_l} \cdot \frac{\lambda_{l} \cdot C_{v,t} \cdot v_{th} \cdot v_{ls}}{g \cdot D_p} + 2 \cdot \frac{\rho_m}{\rho_l} = 2 \cdot \frac{\rho_m}{\rho_l} \cdot \left( \frac{\lambda_{l} \cdot C_{v_t} \cdot v_{th} \cdot v_{ls}}{g \cdot D_p} + 1 \right) \]

So the mixture density is:

\[ \rho_m = \rho_l \cdot \frac{\left( i_{m,a} - i_{m,d} \right)}{2 \cdot \left( \frac{\lambda_{l} \cdot C_{v,t} \cdot v_{th} \cdot v_{ls}}{g \cdot D_p} + 1 \right)} \]

This gives for the delivered volumetric concentration \( C_{vt} \):

\[ \rho_l \cdot \left( 1 + R_{sd} \cdot C_{vt} \right) = \rho_l \cdot \frac{\left( i_{m,a} - i_{m,d} \right)}{2 \cdot \left( \frac{\lambda_{l} \cdot C_{v,t} \cdot v_{th} \cdot v_{ls}}{g \cdot D_p} + 1 \right)} \]

\[ C_{vt} = \frac{1}{R_{sd}} \cdot \left( \frac{\left( i_{m,a} - i_{m,d} \right)}{2} - 1 \right) \]

Again if the terminal hindered settling velocity is very small this can be simplified to:

\[ C_{vt} = \frac{1}{R_{sd}} \cdot \left( \frac{\left( i_{m,a} - i_{m,d} \right)}{2} - 1 \right) \]

This equation is often used to determine the delivered volumetric concentration \( C_{vt} \). For larger particles with a significant terminal hindered settling velocity however, this equation overestimates the delivered volumetric concentration. The
error increases with increasing delivered volumetric concentration, terminal hindered settling velocity and line speed and decreases with increasing pipe diameter. Equation (6.28-56) results in a second degree polynomial and can be solved with the well-known ABC equation.

Now if the spatial volumetric concentration in a horizontal pipe is determined with for example a well calibrated nuclear density meter, the delivered volumetric concentration can be determined with a U Tube. Knowing both concentrations, the so called slip velocity can be determined. Using equation (6.28-56) will give a more accurate estimate of the delivered volumetric concentration and thus of the slip velocity.

6.28.12 Conclusions

After adding the potential energy terms to the hydraulic gradient in a correct way, the pipe inclination effect can be taken into account by multiplying the solids effect term with the cosine of the inclination angle to a power ranging from 1.0 to 1.7. Different researchers give different powers, most probably because the models are either empirical or have different physical backgrounds. This implies that the solids effect reduces to zero for a vertical pipe, which is doubtful, especially for very small particles giving homogeneous flow (ELM). One would expect an equation of the following form:

\[
\overline{i}_{m, \theta} = \overline{i}_l \cdot (1 + \alpha \cdot R_{sd} \cdot C_{vs} \cdot \sin(\theta)^{\beta_1}) + \overline{E}_{rh g} \cdot R_{sd} \cdot C_{vs} \cdot \cos(\theta)^{\beta_2} + (1 + R_{sd} \cdot C_{vs}) \cdot \sin(\theta)
\]

The first term on the right hand side is the Darcy Weisbach friction, including the mobilized ELM (the homogeneous solids effect) corrected for the inclination angle. The second term is the heterogeneous solids effect corrected for the inclination angle. The third term is the potential energy term. So where the heterogeneous solids effect decreases with the inclination angle, the homogeneous solids effect increases. In this form a vertical pipe shows mobilized/reduced ELM behavior, which is observed by Newitt et al. (1961).

The LSDV increases with increasing inclination angle to a maximum with 25%-30% increase for an inclination angle of 15 to 30 degrees. This probably depends on the spatial volumetric concentration, the particle size, the relative submerged density and the particle slip, but not enough data could be found to quantify this.

6.28.13 Nomenclature Inclined Pipes

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, A_p</td>
<td>Cross section pipe</td>
<td>m^2</td>
</tr>
<tr>
<td>A_1</td>
<td>Cross section restricted area above the bed</td>
<td>m^2</td>
</tr>
<tr>
<td>A_2</td>
<td>Cross section bed</td>
<td>m^2</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>--------------------------------------------------</td>
<td>----------</td>
</tr>
<tr>
<td>( C_{vb} )</td>
<td>Bed volumetric concentration</td>
<td>-</td>
</tr>
<tr>
<td>( C_{vs} )</td>
<td>Spatial volumetric concentration</td>
<td>-</td>
</tr>
<tr>
<td>( C_{vt} )</td>
<td>Delivered volumetric concentration</td>
<td>-</td>
</tr>
<tr>
<td>( C_{x} )</td>
<td>Inverse particle Froude number</td>
<td>-</td>
</tr>
<tr>
<td>( D_p )</td>
<td>Pipe diameter</td>
<td>m</td>
</tr>
<tr>
<td>( g )</td>
<td>Gravitational constant (9.81)</td>
<td>m/s²</td>
</tr>
<tr>
<td>( i_l )</td>
<td>Hydraulic gradient liquid without pipe inclination</td>
<td>-</td>
</tr>
<tr>
<td>( i_{l,\theta} )</td>
<td>Hydraulic gradient liquid with pipe inclination</td>
<td>-</td>
</tr>
<tr>
<td>( i_m )</td>
<td>Hydraulic gradient mixture without pipe inclination</td>
<td>-</td>
</tr>
<tr>
<td>( i_{m,\theta} )</td>
<td>Hydraulic gradient mixture with pipe inclination</td>
<td>-</td>
</tr>
<tr>
<td>( L )</td>
<td>Length of pipe</td>
<td>m</td>
</tr>
<tr>
<td>( M )</td>
<td>Wilson heterogeneous power (0.25-1.7)</td>
<td>-</td>
</tr>
<tr>
<td>( O_1 )</td>
<td>Circumference restricted area above the bed in contact with pipe wall</td>
<td>m</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>Circumference of bed with pipe wall</td>
<td>m</td>
</tr>
<tr>
<td>( O_{12} )</td>
<td>Width of the top of the bed</td>
<td>m</td>
</tr>
<tr>
<td>( p )</td>
<td>Pressure in pipe</td>
<td>kPa</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>$R_{sd}$</td>
<td>Relative submerged density of solids</td>
<td>-</td>
</tr>
<tr>
<td>$v_{ls}$</td>
<td>Line speed</td>
<td>m/s</td>
</tr>
<tr>
<td>$v_{50}$</td>
<td>50% stratification velocity</td>
<td>m/s</td>
</tr>
<tr>
<td>$W_b$</td>
<td>Weight of the bed</td>
<td>ton</td>
</tr>
<tr>
<td>$W_{b,s}$</td>
<td>Submerged weight of the bed</td>
<td>ton</td>
</tr>
<tr>
<td>$x$</td>
<td>Distance in pipe length direction</td>
<td>m</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>Density of the bed including pore water</td>
<td>ton/m$^3$</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>Density of the solids</td>
<td>ton/m$^3$</td>
</tr>
<tr>
<td>$\rho_l$</td>
<td>Density of the liquid</td>
<td>ton/m$^3$</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>Mixture density</td>
<td>ton/m$^3$</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>Shear stress between liquid and pipe wall</td>
<td>kPa</td>
</tr>
<tr>
<td>$\tau_{12}$</td>
<td>Shear stress on top of the bed</td>
<td>kPa</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Inclination angle (positive upwards, negative downwards)</td>
<td>°</td>
</tr>
<tr>
<td>$\mu_{sf}$</td>
<td>Sliding friction coefficient</td>
<td>-</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Durand ordinate</td>
<td>-</td>
</tr>
</tbody>
</table>