4.10: The Laplacian Operator

The Laplacian \( \nabla^2 f \) of a field \( f({\bf r}) \) is the divergence of the gradient of that field:

\[
\nabla^2 f \triangleq \nabla \cdot \left( \nabla f \right) \quad \label{m0099_eLaplaceDef}
\]

Note that the Laplacian is essentially a definition of the second derivative with respect to the three spatial dimensions. For example, in Cartesian coordinates,

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]

as can be readily verified by applying the definitions of gradient and divergence in Cartesian coordinates to Equation \ref{m0099_eLaplaceDef}.

The Laplacian relates the electric potential (i.e., \( V \), units of V) to electric charge density (i.e., \( \rho_v \), units of C/m\(^3\)). This relationship is known as Poisson’s Equation:

\[
\nabla^2 V = - \frac{\rho_v}{\epsilon} \quad \nonumber
\]

where \( \epsilon \) is the permittivity of the medium. The fact that \( V \) is related to \( \rho_v \) in this way should not be surprising, since electric field intensity \( \nabla (\nabla \cdot {\bf E}) \), units of V/m, is proportional to the derivative of \( V \) with respect to distance (via the gradient) and \( \nabla \cdot {\bf E} \) is proportional to the derivative of \( \nabla \cdot {\bf E} \) with respect to distance (via the divergence).

The Laplacian operator can also be applied to vector fields; for example, Equation \ref{m0099_eLaplaceScalar} is valid even if the scalar field \( f \) is replaced with a vector field. In the Cartesian coordinate system, the Laplacian of the vector field \( \nabla \cdot ({\bf A}) = \hat{\bf x}A_x + \hat{\bf y}A_y + \hat{\bf z}A_z \) is
An important application of the Laplacian operator of vector fields is the wave equation; e.g., the wave equation for $\{\textbf{E}\}$ in a lossless and source-free region is

$$\nabla^2 \textbf{E} + \beta^2 \textbf{E} = 0$$

where $\beta$ is the phase propagation constant.

It is sometimes useful to know that the Laplacian of a vector field can be expressed in terms of the gradient, divergence, and curl as follows:

$$\nabla^2 \{\textbf{A}\} = \nabla(\nabla \cdot \{\textbf{A}\}) - \nabla \times (\nabla \times \{\textbf{A}\})$$

The Laplacian operator in the cylindrical and spherical coordinate systems is given in Appendix B2.