8.5: Derivations of the Momentum Equation

Fig. 8.8 The shear stress at different surfaces. All shear stress shown in surface \((x')\) and \((x+dx')\).

Previously it was shown that equation (??) is equivalent to Newton second law for fluids. Equation (??) is also applicable for the small infinitesimal cubic. One direction of the vector equation will be derived for \((x)\) Cartesian coordinate (see Figure 8.8). Later Newton second law will be used and generalized. For surface forces that acting on the cubic are surface forces, gravitation forces (body forces), and internal forces. The body force that acting on infinitesimal cubic in \((x)\) direction is

\[ \label{dif:eq:momentumG} \]
The surface forces in \(x\) direction on the \(y\) surface on are

\[
f_{xy} = \left. \tau_{yx} \right|_{y+dy} \times \overbrace{dx\,dz}^{dA_y} - \left. \tau_{yx} \right|_{y} \times \overbrace{dx\,dz}^{dA_y}
\]

The same can be written for the \(z\) direction. The shear stresses can be expanded into Taylor series as

\[
\left. \tau_{ix} \right|_{i+di} = \tau_{ix} + \left. \frac{\partial \left( \tau_{ix} \right) }{\partial i} \right|_{i} di + \cdots
\]

where \(i\) in this case is \(x\), \(y\), or \(z\). Hence, the total net surface force results from the shear stress in the \(x\) direction is

\[
f_x = \left( \frac{\partial \tau_{xx} }{\partial x\dfrac{}{}} + \frac{\partial \tau_{yx} }{\partial y} + \frac{\partial \tau_{zx} }{\partial z} \right) dx\,dy\,dz
\]

after rearrangement equations such as (2) and (3) transformed into equivalent equation (6) for \(y\) coordinate is

\[
f_y = \left( \frac{\partial \tau_{yy} }{\partial y\dfrac{}{}} + \frac{\partial \tau_{yx} }{\partial x} + \frac{\partial \tau_{zy} }{\partial z} \right) dy\,dx\,dz
\]

\[
f_z = \left( \frac{\partial \tau_{zz} }{\partial z\dfrac{}{}} + \frac{\partial \tau_{xz} }{\partial x} + \frac{\partial \tau_{zy} }{\partial y} \right) dz\,dx\,dy
\]
\[
\rho \frac{DU_y}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho \mathbf{f}_y
\]
The same can be obtained for the \(z\) component

\[
\rho \frac{DU_z}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho \mathbf{f}_z
\]

\[
\rho \frac{DU_i}{Dt} = \frac{\partial \tau_{ii}}{\partial i} + \frac{\partial \tau_{ji}}{\partial j} + \frac{\partial \tau_{ki}}{\partial j} + \rho \mathbf{f}_i
\]

Where \(i\) is the balance direction and \(j\) and \(k\) are two other coordinates. Equation (9) can be written in a vector form which combined all three components into one equation. The advantage of the vector form allows the usage of the different coordinates. The vector form is

\[
\rho \frac{D\pmb{U}}{Dt} = \nabla \cdot \pmb{\tau}^{(i)} + \rho \pmb{f}_G
\]

\[
\pmb{\tau}^{(i)} = \tau_{ix} \hat{i} + \tau_{iy} \hat{j} + \tau_{iz} \hat{k}
\]
\( \] is part of the shear stress tensor and \( (i) \) can be any of the \( (x, y, z) \) or \( (z') \). Or in index (Einstein) notation as

\[
\rho \left( \frac{DU_i}{Dt} \right) = \frac{\partial \tau_{ji}}{\partial x_i} + \rho \left( f_{Gi} \right)
\]

Equations (6) or (7) or (8) requires that the stress tensor be defined in term of the velocity/deformation. The relationship between the stress tensor and deformation depends on the classes of materials the stresses acts on. Additionally, the deformation can be viewed as a function of the velocity field. As engineers do in general, the simplest model is assumed which referred as the solid continuum model. In this model the relationship between the (shear) stresses and rate of strains are assumed to be linear. In solid material, the shear stress yields a fix amount of deformation. In contrast, when applying the shear stress in fluids, the result is a continuous deformation. Furthermore, reduction of the shear stress does not return the material to its original state as in solids. The similarity to solids the increase shear stress in fluids yields larger deformations. Thus this "solid" model is a linear relationship with three main assumptions:

1. There is no preference in the orientation (also call isentropic fluid),
2. there is no left over stresses (In other words when the "no shear stress" situation exist the rate of deformation or strain is zero), and
3. a linear relationship exist between the shear stress and the rate of shear strain. At time \( \text{at} \), the control volume is at a square shape and at a location as depicted in Figure 8.9 (by the blue color). At time \( \text{at} + dt \) the control volume undergoes three different changes. The control volume moves to a new location, rotates and changes the shape (the purple color in in Figure 8.9). The translational movement is referred to a movement of body without change of the body and without rotation. The rotation is the second movement that referred to a change in of the relative orientation inside

\[ Fig. 8.9 Control volume at \( \text{at} \) and \( \text{at} + dt \) under continuous angle deformation. Notice the three combinations of the deformation shown by purple color relative to blue color. \]

1. There is no preference in the orientation (also call isentropic fluid),
2. there is no left over stresses (In other words when the "no shear stress" situation exist the rate of deformation or strain is zero), and
3. a linear relationship exist between the shear stress and the rate of shear strain. At time \( \text{at} \), the control volume is at a square shape and at a location as depicted in Figure 8.9 (by the blue color). At time \( \text{at} + dt \) the control volume undergoes three different changes. The control volume moves to a new location, rotates and changes the shape (the purple color in in Figure 8.9). The translational movement is referred to a movement of body without change of the body and without rotation. The rotation is the second movement that referred to a change in of the relative orientation inside
the control volume. The third change is the misconfiguration or control volume (deformation). The deformation of the control volume has several components (see the top of Figure 8.9). The shear stress is related to the change in angle of the control volume lower left corner. The angle between \( \gamma(x) \) to the new location of the control volume can be approximate for a small angle as

\[
\frac{d\gamma_x}{dt} = \tan\left( \frac{U_y + \frac{dU_y}{dx}dx - U_y}{dx} \right) = \tan\left( \frac{dU_y}{dx} \right) \approx \frac{dU_y}{dx}
\]

In these derivatives, the symmetry \( \frac{dU_y}{dx} \neq \frac{dU_x}{dy} \) was not assumed and or required because rotation of the control volume. However, under isentropic material it is assumed that all the shear stresses contribute equally. For the assumption of a linear fluid.

\[
\frac{d\gamma_{xy}}{Dt} = \frac{dU_y}{dx} + \frac{dU_x}{dy}
\]

where, \( \mu \) is the "normal" or "ordinary" viscosity coefficient which relates the linear coefficient of proportionality and shear stress. This deformation angle coefficient is assumed to be a property of the fluid. In a similar fashion it can be written to other directions for \( \gamma(x, z) \) as

\[
\frac{d\gamma_{xz}}{Dt} = \frac{dU_z}{dx} + \frac{dU_x}{dz}
\]

and for the directions of \( \gamma(y, z) \) as

\[
\frac{d\gamma_{yz}}{Dt} = \frac{dU_z}{dy} + \frac{dU_y}{dz}
\]

is assumed to be the same regardless of the direction. This assumption is referred as isotropic viscosity. It can be noticed at this stage, the relationship for the two of stress tensor parts was established. The only missing thing, at this stage, is the diagonal component which to be dealt below.

In general equation (15) can be written as

\[
\frac{d\gamma_{ij}}{Dt} = \mu \left( \frac{dU_j}{di} + \frac{dU_i}{dj} \right)
\]
Normal Stress

The normal stress, \(\tau_{ii}\) (where \(i\) is either \(x\), \(y\), \(z\)) appears in the shear matrix diagonal. To find the main (or the diagonal) stress the coordinates are rotate by \(45^\circ\). The diagonal lines (line and line in Figure) in the control volume move to the new locations. In addition, the sides rotate in unequal amount which make one diagonal line longer and one diagonal line shorter. The normal shear stress relates to the change in the diagonal line length change. This relationship can be obtained by changing the coordinates orientation as depicted by Figure 8.10. The \(\delta dx\) is constructed so it equals to \(\delta dy\). The forces acting in the direction of \(\delta x\) on the element are combination of several terms. For example, on the \(\delta x\) surface (lower surface) and the \(\delta y\) (left) surface, the shear stresses are acting in this direction. It can be noticed that \(\delta (d x')\) surface is \(\sqrt{2}\) times larger than \(\delta dx\) and \(\delta dy\) surfaces. The force balance in the \(\delta x\) is

\[
\overbrace{dy}^{A_x} \tau_{xx} \overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_x} + \overbrace{dx}^{A_y} \tau_{yy} \overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dx}^{A_y} \tau_{yx} \overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dy}^{A_x} \tau_{xy} \overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} = \overbrace{dx\sqrt{2}}^{A_{x'}} \tau_{x'x'}
\]

Dividing by \(\delta dx\) and after some rearrangements utilizing the identity \(\tau_{xy} = \tau_{yx}\) results in

\[
\frac{\tau_{xx} + \tau_{yy}}{2} + \tau_{yx} = \tau_{x'x'}
\]

Setting the similar analysis in the \(\delta y\) results in

\[
\frac{\tau_{xx} + \tau_{yy}}{2} - \tau_{yx} = \tau_{y'y'}
\]

Subtracting (21) from (20) results in

\[
2\tau_{yx} = \tau_{x'x'} - \tau_{y'y'}
\]
Figure 8.11 Different triangles deformation for the calculations of the normal stress.

or dividing by \( (2! \) equation (22) becomes

\[
\tau_{yx} = \frac{1}{2} \left( \tau_{x' x'} - \tau_{y' y'} \right)
\]

Equation (22) relates the difference between the normal shear stress and the normal shear stresses in \((x', y')\) coordinates) and the angular strain rate in the regular \((x, y)\) coordinates). The linear deformations in the \((x', y')\) directions which is rotated \(45^\circ\) relative to the \((x, y)\) axes can be expressed in both coordinates system. The angular strain rate in the \((x, y)\) frame is related to the strain rates in the \((x', y')\) frame. Figure 8.11(a) depicts the deformations of the triangular particles between time \(t\) and \(t+dt\). The small deformations a, b, c, and d in the Figure are related to the incremental linear strains. The rate of strain in the \((x')\) direction is

\[
d\epsilon_x = \frac{c}{dx}
\]

The rate of the strain in \((y')\) direction is

\[
d\epsilon_y = \frac{a}{dx}
\]

The total change in the deformation angle is related to \(\tan(\theta)\), in both sides \((d/dx + b/dy)\) which in turn is related to combination of the two sides angles. The linear angular deformation in \((xy)\) direction is

\[
d\gamma_{xy} = \frac{b+d}{dx}
\]

Here, \((d\epsilon_x, d\epsilon_y)\) is the linear strain (increase in length divided by length) of the particle in the \((x')\) direction, and \((d\epsilon_{x'}, d\epsilon_{y'})\) is its linear strain in the \(y\)-direction. The linear strain in the \((x')\) direction can be computed by observing Figure 8.11(b) The hypotenuse of the triangle is oriented in the \((x, x')\) direction (again observe Figure 8.11(b)). The original length of the hypotenuse \(\sqrt{2} \times dx\). The change in the hypotenuse length is \(\sqrt{(c+b)^2 + (a+d)^2}\). It can be approximated that the change is about \(45^\circ\) because changes are infinitesimally small. Thus, \(\cos 45^\circ\) or \(\sin 45^\circ\) times the change contribute as first approximation to change. Hence, the ratio strain in the \((x')\) direction is

\[
\]}
\[ d\epsilon_{x'} = \frac{\sqrt{(c+b)^2 + (a+d)^2}}{\sqrt{2}dx} \simeq \frac{\frac{(c+b)}{\sqrt{2}} + \frac{(c+b)}{\sqrt{2}} + \frac{f (dx')}{\sim 0}}{\sqrt{2}dx} \]\n
Equation (27) can be interpreted as (using equations (24), (25), and (26)) as

\[ d\epsilon_{x'} = \frac{1}{2} \left( \frac{a+b+c+d}{dx} \right) = \frac{1}{2} \left( d\epsilon_y + d\epsilon_y + d\gamma_{xy} \right) \]

In the same fashion, the strain in \( y' \) coordinate can be interpreted to be

\[ d\epsilon_{y'} = \frac{1}{2} \left( d\epsilon_y + d\epsilon_y - d\gamma_{xy} \right) \]

Notice the negative sign before \( d\gamma_{xy} \). Combining equation (28) with equation (29) results in

\[ d\epsilon_{x'} - d\epsilon_{y'} = d\gamma_{xy} \]

Equation (30) describing in Lagrangian coordinates a single particle. Changing it to the Eulerian coordinates transforms the equation into

\[ \frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{D\gamma_{xy}}{Dt} \]

From (15) it can be observed that the right hand side of equation (31) can be replaced by \( \frac{\tau_{xy}}{\mu} \) to read

\[ \frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{\tau_{xy}}{\mu} \]

From equation (22) \( \tau_{xy} \) be substituted and equation (32) can be continued and replaced as

\[ \frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{1}{2\mu} \left( \tau_{x'x'} - \tau_{y'y'} \right) \]
Fig. 8.12 Linear strain of the element purple denotes \((t)\) and blue is for \((t+dt)\). Dashed squares denotes the movement without the linear change.

Figure 8.12 depicts the approximate linear deformation of the element. The linear deformation is the difference between the two sides as

\[
\frac{D \epsilon_{x'}}{Dt} = \frac{\partial U_{x'}}{\partial x'}
\]

The same way it can be written for the \((y')\) coordinate.

\[
\frac{D \epsilon_{y'}}{Dt} = \frac{\partial U_{y'}}{\partial y'}
\]

Equation (34) can be written in the \((y'\text{ Kern-1pt lower+1pt hbox{}}')\) coordinate and is similar by substituting the coordinates.

The rate of strain relations can be substituted by the velocity and equations (34) and (35) changes into

\[
\tau_{x'x'} - \tau_{y'y'} = 2 \mu \left( \frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{y'}}{\partial y'} \right)
\]

Similar two equations can be obtained in the other two plans. For example in \((y'\text{ Kern-1pt lower+1pt hbox{}}')\) plan one can obtained

\[
\tau_{x'x'} - \tau_{y'y'}
\]
\[ \tau_{z'z'} = 2 \mu \left( \frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{z'}}{\partial z'} \right) \]

Adding equations (36) and (37) results in

\[ \label{dif:eq:totalStressDefraction} \]
\[ \overbrace{\left( 3 - 1 \right)}^{2} \tau_{x'x'} - \tau_{y'y'} - \tau_{z'z'} = \overbrace{\left( 6 - 2 \right)}^{4} \mu \frac{\partial U_{x'}}{\partial x'} - 2 \mu \left( \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \]

Rearranging equation (38) transforms it into

\[ \label{dif:eq:totalStressDefraction1} \]
\[ 3 \tau_{x'x'} = \tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'} + 6 \mu \frac{\partial U_{x'}}{\partial x'} - 2 \mu \left( \frac{\partial U_{x'}}{\partial x'} + \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \]

Dividing the results by 3 so that one can obtained the following

\[ \label{dif:eq:totalStressDefraction2} \]
\[ \tau_{x'x'} = \overbrace{\frac{\tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'}}{3}} \]
\[ \overbrace{\left( 3 \right)} \]
The "mechanical" pressure, $P_m$, is defined as the (negative) average value of pressure in directions of $(x')-(y')-(z')$. This pressure is a true scalar value of the flow field since the property is averaged invariant to the coordinate transformation. In situations where the main diagonal terms of the stress tensor are not the same in all directions (in some viscous flows) this property can be served as a measure of the local normal stress. The mechanical pressure can be defined as averaging of the normal stress acting on a infinitesimal sphere. It can be shown that this two definitions are "identical" in the limits. With this definition and noticing that the coordinate system $(x')-(y')-(z')$ has no special significance and hence equation (40) must be valid in any coordinate system thus equation (40) can be written as

\[
\begin{align*}
\tau_{xx} &= - P_m + 2 \mu \frac{\partial U_x}{\partial x} + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \\
\tau_{xy} &= - \left( P_m + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \right) + \mu \left( \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right)
\end{align*}
\]

Again where $(P_m)$ is the mechanical pressure and is defined as

Mechanical Pressure

\[
\begin{align*}
P_m &= - \frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3}
\end{align*}
\]

It can be observed that the non main (diagonal) terms of the stress tensor are represented by an equation like (18). Commonality engineers like to combined the two difference expressions into one as

\[
\begin{align*}
\tau_{xy} &= - \left( P_m + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \right) \delta_{xy} + \mu \left( \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right)
\end{align*}
\]

or
\[ \tau_{xx} = - \left( P_m + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \right) \delta_{xy} + \mu \left( \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right) \]

**Advance Material**

or index notation

\[ \tau_{ij} = - \left( P_m + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \right) \delta_{ij} + \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \]

**End Advance Material**

where \( \delta_{ij} \) is the Kronecker delta what is \( \delta_{ij} = 1 \) when \( i=j \) and \( \delta_{ij} = 0 \) otherwise. While this expression has the advantage of compact writing, it does not add any additional information. This expression suggests a new definition of the thermodynamical pressure is

**Thermodynamic Pressure**

\[ P = P_m + \frac{2}{3} \mu \nabla \cdot \mathbf{U} \]

**Summary of the Stress Tensor**

The above derivations were provided as a long mathematical explanation. To reduced one unknown (the shear stress) equation (6) the relationship between the stress tensor and the velocity were to be established. First, connection between \( \tau_{xy} \) and the deformation was built. Then the association between normal stress and perpendicular stress was constructed. Using the coordinates transformation, this association was established. The linkage between the stress in the rotated coordinates to the deformation was established.

**Second Viscosity Coefficient**

The coefficient \( \frac{2}{3} \mu \) is experimental and relates to viscosity. However, if the derivations before were to include additional terms, an additional correction will be needed. This correction results in

\[ P = P_m + \lambda \nabla \cdot \mathbf{U} \]

The value of \( \lambda \) is obtained experimentally. This coefficient is referred in the literature by several terms such as the "expansion viscosity" ",second coefficient of viscosity" and "bulk viscosity." Here the term bulk viscosity will be adapted. The dimension of the bulk viscosity, \( /\lambda / \), is similar to the viscosity \( /\mu / \). According to second law of...
thermodynamic derivations (not shown here and are under construction) demonstrate that \( \lambda \) must be positive. The thermodynamic pressure always tends to follow the mechanical pressure during a change. The expansion rate of change and the fluid molecular structure through \( \lambda \) control the difference. Equation (47) can be written in terms of the thermodynamic pressure \( P \), as

\[
\begin{align*}
\tau_{ij} &= - \left[ P + \left( \frac{2}{3} \mu - \lambda \right) \nabla \cdot \mathbf{U} \right] \delta_{ij} \\
+ \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)
\end{align*}
\]  

The significance of the difference between the thermodynamic pressure and the mechanical pressure associated with fluid dilation which connected by \( \nabla \cdot \mathbf{U} \). The physical meaning of \( \nabla \cdot \mathbf{U} \) represents the relative volume rate of change. For simple gas (dilute monatomic gases) it can be shown that \( \lambda \) vanishes. In material such as water, \( \lambda \) is large (3 times \( \mu \)) but the net effect is small because in that cases \( \nabla \cdot \mathbf{U} \rightarrow 0 \). For complex liquids this coefficient, \( \lambda \), can be over 100 times larger than \( \mu \). Clearly for incompressible flow, the whole effect is vanished. In most cases, the total effect of the dilation on the flow is very small. Only in micro fluids and small and molecular scale such as in shock waves this effect has some significance. In fact this effect is so insignificant that there is difficulty in to construct experiments so this effect can be measured. Thus, neglecting this effect results in

\[
\begin{align*}
\tau_{ij} &= - P \delta_{ij} \\
+ \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)
\end{align*}
\]  

To explain equation (49), it can be written for specific coordinates. For example, for the \( \tau_{xx} \) it can be written that

\[
\begin{align*}
\tau_{xx} &= - P + 2 \frac{\partial U_x}{\partial x} \\
\end{align*}
\]  

and the \( y \) coordinate the equation is

\[
\begin{align*}
\tau_{yy} &= - P + 2 \frac{\partial U_y}{\partial y} \\
\end{align*}
\]  

However the mix stress, \( \tau_{xy} \), is

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} = \left( \frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y} \right)
\end{align*}
\]  

For the total effect, substitute equation (48) into equation (6) which results in

\[
\begin{align*}
\rho \left( \frac{D U_x}{Dt} \right) &= -\frac{\partial }{\partial x} \left[ P + \left( \frac{2}{3} \mu - \lambda \right) \nabla \cdot \mathbf{U} \right] \\
+ \mu \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right)
\end{align*}
\]
\[ + \{\text{pmb}(f)_{\text{B}}\}_x \]
\end{array}
\]

or in a vector form as

N-S in stationary Coordinates

\[
\begin{array}{c}
\text{\label{dif:eq:nsGv}}
\rho \ \dfrac{D \ \text{pmb}(U)}{Dt} \\
= - \nabla P + \left(\dfrac{1}{3} \mu + \lambda \right) \nabla \cdot \text{pmb}(U) + \mu \nabla^2 \text{pmb}(U) \quad \text{or in a vector form as} \quad \text{\label{dif:eq:nsGvIncompressibleFlow1}}
\end{array}
\]

For index form as

\[
\begin{array}{c}
\text{\label{dif:eq:nsG}}
\rho \ \dfrac{D \ U_i}{Dt} \\
= - \dfrac{\partial }{\partial x_i} \left( P+ \left(\dfrac{2}{3} \mu - \lambda \right) \nabla \cdot \text{pmb}(U) \right) + \mu \dfrac { \partial^2 \text{pmb}(U) }{\partial x_i \partial x_j} \quad \text{\label{dif:eq:nsGIncompressibleFlow}}
\end{array}
\]

For incompressible flow the term \(\nabla \cdot \text{pmb}(U)\) vanishes, thus equation (54) is reduced to

Momentum for Incompressible Flow

\[
\begin{array}{c}
\text{\label{dif:eq:nsGvlncmpsrbleFlow1}}
\rho \ \dfrac{D \ \text{pmb}(U)}{Dt} \\
= - \nabla P + \mu \nabla^2 \text{pmb}(U) \quad \text{\label{dif:eq:momEqx}}
\end{array}
\]

or in the index notation it is written

\[
\begin{array}{c}
\text{\label{dif:eq:nsGvlncmpsrbleFlow}}
\rho \ \dfrac{D \ U_i}{Dt} \\
= - \dfrac{\partial P }{\partial x_i} + \mu \dfrac { \partial^2 \text{pmb}(U) }{\partial x_i \partial x_j} \quad \text{\label{dif:eq:momEqx}}
\end{array}
\]

The momentum equation in Cartesian coordinate can be written explicitly for \(x\) coordinate as
\[g_x\] is the the body force in the \(x\) direction. In the \((y)\) coordinate the momentum equation is

\[ \rho \left( \frac{\partial U_y}{\partial t} + U_x \frac{\partial U_y}{\partial x} + U_y \frac{\partial U_y}{\partial y} + U_z \frac{\partial U_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y \]

in \((z)\) coordinate the momentum equation is

\[ \rho \left( \frac{\partial U_z}{\partial t} + U_x \frac{\partial U_z}{\partial x} + U_y \frac{\partial U_z}{\partial y} + U_z \frac{\partial U_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2} \right) + \rho g_z \]

---

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