8.7: Examples for Differential Equation (Navier-Stokes)

Examples of an one-dimensional flow driven by the shear stress and pressure are presented. For further enhance the understanding some of the derivations are repeated. First, example dealing with one phase are present. Later, examples with two phase are presented.

Example 8.6

Incompressible liquid flows between two infinite plates from the left to the right (as shown in Figure 8.16) The distance between the plates is \(\ell\). The static pressure per length is given as \(\Delta P\). The upper surface is moving in velocity, \(U_{\ell}\) (The right side is defined as positive).

Solution 8.6

In this example, the mass conservation yields

\[
\text{\ref{dif:eq:2dmass}}
\]
\[ \frac{d}{dt} \int_{cv} \rho \, dV \overset{=0}{=} - \int_{cv} \rho \, \dot{U}_{rn} \, dA = 0 \tag{71} \]

The momentum is not accumulated (steady state and constant density). Further because no change of the momentum thus

\[ \int_A \rho \, U_x \, \dot{U}_{rn} \, dA = 0 \tag{72} \]

Thus, the flow in and the flow out are equal. It can concluded that the velocity in and out are the same (for constant density). The momentum conservation leads

\[ - \int_{cv} \boldsymbol{P} \, dA + \int_{cv} \boldsymbol{\tau}_{xy} \, dA = 0 \tag{73} \]

The reaction of the shear stress on the lower surface of control volume based on Newtonian fluid is

\[ \boldsymbol{\tau}_{xy} = - \mu \, \left( \frac{dU}{dy} \right) \tag{74} \]

On the upper surface is different by Taylor explanation as

\[ \boldsymbol{\tau}_{xy} = \mu \left( \frac{dU}{dy} + \frac{d^2U}{dy^2} \, dy + \overset{\text{cong 0}}{\overbrace{\frac{d^3U}{dy^3} \, dy^2 + \cdots}} \right) \tag{75} \]

The net effect of these two will be difference between them

\[ \mu \left( \frac{dU}{dy} \right) - \frac{dU}{dy} \cong \mu \, \frac{d^2U}{dy^2} \, dy \tag{76} \]

The assumptions is that there is no pressure difference in the \((z)\) direction. The only difference in the pressure is in the \((x)\) direction and thus

\[ P - \left( P + \frac{dP}{dx} \, dx \right) = -\frac{dP}{dx} \, dx \tag{77} \]

A discussion why \(\frac{\partial P}{\partial y} \sim 0\) will be presented later. The momentum equation in the \((x)\) direction (or from equation (58)) results (without gravity effects) in

\[ -\frac{dP}{dx} = \mu \left( \frac{d^2U}{dy^2} \right) \tag{78} \]
Equation (78) was constructed under several assumptions which include the direction of the flow, Newtonian fluid. No assumption was imposed on the pressure distribution. Equation (78) is a partial differential equation but can be treated as ordinary differential equation in the \( z \) direction of the pressure difference is uniform. In that case, the left hand side is equal to constant. The "standard" boundary conditions is non-vanishing pressure gradient (that is the pressure exist) and velocity of the upper or lower surface or both. It is common to assume that the "no slip" condition on the boundaries condition. The boundaries conditions are

\[
\begin{align*}
U_x(y=0) &= 0 \quad \tag{79} \\
U_x(y=\ell) &= U_{\ell} \quad \tag{80}
\end{align*}
\]

The solution of the "ordinary" differential equation (78) after the integration becomes

\[
\begin{align*}
U_x &= -\frac{1}{2} \frac{dP}{dx} y^2 + c_2 y + c_3 \quad \tag{81}
\end{align*}
\]

Applying the boundary conditions, equation (79) results in

\[
\begin{align*}
U_x(y) &= \frac{y}{\ell} \left( \frac{\ell^2}{2 U_0 \mu} \frac{dP}{dx} \left( 1 - \frac{y}{\ell} \right) \right) + \frac{y}{\ell} \quad \tag{82}
\end{align*}
\]
For the case where the pressure gradient is zero the velocity is linear as was discussed earlier in Chapter 1 (see Figure ??). However, if the plates or the boundary conditions do not move the solution is

\[
U_x(y) = \left( \frac{\ell^2}{U_0 \mu \frac{dP}{dx}} \left( 1 - \frac{y}{\ell} \right) \right) + \frac{y}{\ell} \tag{83}
\]

What happens when \(\frac{\partial P}{\partial y} \sim 0\)?

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**Fig. 8.18 The control volume of liquid element in cylindrical coordinates.**

**Cylindrical Coordinates**

Similarly the problem of one dimensional flow can be constructed for cylindrical coordinates. The problem is still one dimensional because the flow velocity is a function of (only) radius. This flow referred as Poiseuille flow after Jean Louis Poiseuille a French Physician who investigated blood flow in veins. Thus, Poiseuille studied the flow in a small diameters (he was not familiar with the concept of Reynolds numbers). Rederivation are carried out for a short cut. The momentum equation for the control volume depicted in the Figure 8.18 is

\[
- \int \pmb{P} \, dA + \int \boldsymbol{\tau} \, dA = \int \rho \, U_z \, U_{rn} \, dA \tag{84}
\]

The shear stress in the front and back surfaces do no act in the \(\phi\) direction. The shear stress on the circumferential part small dark blue shown in Figure 8.18 is

\[
\int \boldsymbol{\tau} \, dA = \mu \left( \frac{dU_z}{dr} \right) \overbrace{2\,\pi\,r\,dz}^{dA} \tag{85}
\]

The pressure integral is

\[
\int \pmb{P} \, dA = \int \left( P_{z_dz} - P_z \right) \pi r^2 = \]

\[
\]

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\[
\left( P_z + \dfrac{\partial P}{\partial z} \right) \pi r^2 = \dfrac{\partial P}{\partial z} \pi r^2 \tag{86}
\]

The last term is

\[
\begin{array}{rl}
\displaystyle \int \rho \, U_z \, U_{rn} \, dA = & \rho \int U_z \, U_{rn} \, dA = \\
\displaystyle \rho \left( \int_{z+dz} \left( U_{z+dz} \right)^2 dA \right) - \left( \int_{z} \left( U_z \right)^2 dA \right) \\
= & \rho \int_{z} \left( \left( U_{z+dz} \right)^2 - \left( U_z \right)^2 \right) dA
\end{array}
\]

The term \( \left( \left( U_{z+dz} \right)^2 - \left( U_z \right)^2 \right) \) is zero because \( \left( U_{z+dz} \right) = \left( U_z \right) \) because mass conservation for any element. Hence, the last term is

\[
\int \rho \, U_z \, U_{rn} \, dA = 0 \tag{87}
\]

Substituting equation (85) and (86) into equation (84) results in

\[
\mu \dfrac{dU_z}{dr} 2\pi r \, dz = - \dfrac{\partial P}{\partial z} \pi r^2 \tag{88}
\]

Which shrinks to

\[
\dfrac{2 \, \mu}{r} \dfrac{dU_z}{dr} = - \dfrac{\partial P}{\partial z} \tag{89}
\]

Equation (89) is a first order differential equation for which only one boundary condition is needed. The "no slip" condition is assumed

\[
U_z(r=R) = 0 \tag{90}
\]

Where \( R \) is the outer radius of pipe or cylinder. Integrating equation (89) results in

\[
U_z = - \dfrac{1}{\mu} \dfrac{\partial P}{\partial z} r^2 + c_1 \tag{91}
\]

was eliminated due to the smart short cut. The integration constant obtained via the application of the boundary condition which is
\[ c_1 = -\frac{1}{\mu} \frac{\partial P}{\partial z} R^2 \tag{92} \]

The solution is
\[
U_z = \frac{1}{\mu} \frac{\partial P}{\partial z} R^2 \left(1 - \left( \frac{r}{R}\right)^2 \right) \tag{93}
\]

While the above analysis provides a solution, it has several deficiencies which include the ability to incorporate different boundary conditions such as flow between concentric cylinders.

Example 8.7

Fig. 8.19 Liquid flow between concentric cylinders for example.

A liquid with a constant density is flowing between concentric cylinders as shown in Figure 8.19. Assume that the velocity at the surface of the cylinders is zero calculate the velocity profile. Build the velocity profile when the flow is one directional and viscosity is Newtonian. Calculate the flow rate for a given pressure gradient.

Solution 8.7

After the previous example, the appropriate version of the Navier–Stokes equation will be used. The situation is best suitable to solved in cylindrical coordinates. One of the solution of this problems is one dimensional solution. In fact there is no physical reason why the flow should be only one dimensional. However, it is possible to satisfy the boundary conditions. It turn out that the "simple" solution is the first mode that appear in reality. In this solution will be discussing the flow first mode. For this mode, the flow is assumed to be one dimensional. That is, the velocity isn't a function of the angle, or \(z\) coordinate. Thus only equation in \(z\) coordinate is needed. It can be noticed that this case is steady state and also the acceleration (convective acceleration) is zero
\[
\rho \left( \overbrace{\frac{\partial U_z}{\partial t}}^{\neq f(t)} + \overbrace{U_r}^{= 0} \frac{\partial U_z}{\partial r} + \overbrace{\frac{U_{\phi}}{r}}^{=0} \frac{\partial U_z}{\partial \phi} + U_z \overbrace{\frac{\partial U_z}{\partial z}}^{=0} \right) = 0 \tag{94}
\]
The steady state governing equation then becomes
\[
\rho \left( \cancel{0} \right) = 0 = -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_z}{\partial r} \right) \right) + \overbrace{\cdots}^{=0} + \cancel{\rho \, g_z} \tag{95}
\]
The PDE above () required boundary conditions which are
\[
U_z(r=r_i) = 0 \\
U_z(r=r_o) = 0
\]
Integrating equation (95) once results in
\[
\frac{r \, \partial U_z}{\partial r} = \frac{1}{2 \, \mu} \frac{\partial P}{\partial z} r^2 + c_1 \tag{96}
\]
Dividing equation (96) and integrating results for the second times results
\[
\frac{\partial U_z}{\partial r} = \frac{1}{2 \, \mu} \frac{\partial P}{\partial z} r + \frac{c_1}{r} \tag{97}
\]
Integration of equation (97) results in
\[
U_z = \frac{1}{4 \, \mu} \frac{\partial P}{\partial z} r^2 + c_1 \ln r + c_2 \tag{98}
\]
Applying the first boundary condition results in
\[
0 = \frac{1}{4 \, \mu} \frac{\partial P}{\partial z} r_i^2 + c_1 \ln r_i + c_2 \tag{99}
\]
applying the second boundary condition yields
\[
0 = \frac{1}{4 \, \mu} \frac{\partial P}{\partial z} r_o^2 + c_1 \ln r_o + c_2 \tag{100}
\]
The solution is
\[
c_1 = \frac{1}{4 \, \mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( r_o^2 - r_i^2 \right) \\
c_2 = \frac{1}{4 \, \mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( r_o^2 - r_i^2 \right)
\]
The solution is when substituting the constants into equation (98) results in
\[
\begin{array}{rcl}
U_z(r) &=& \frac{1}{4\mu} \frac{\partial P}{\partial z} r^2 + \frac{1}{4\mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( r_o^2 - r_i^2 \right) \ln r \\
&+& \frac{1}{4\mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( \ln(r_i) r_o^2 - \ln(r_o) r_i^2 \right)
\end{array}
\tag{101}
\]
The flow rate is then
\[
Q = \int_{r_i}^{r_o} U_z(r) \, dA \tag{102}
\]
Or substituting equation (101) into equation (102) transformed into
\[
Q = \int_A \left[ \frac{1}{4\mu} \frac{\partial P}{\partial z} r^2 + \frac{1}{4\mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( r_o^2 - r_i^2 \right) \ln r \\
+ \frac{1}{4\mu} \ln \left( \frac{r_o}{r_i} \right) \frac{\partial P}{\partial z} \left( \ln(r_i) r_o^2 - \ln(r_o) r_i^2 \right) \right] \, dA
\tag{103}
\]
A finite integration of the last term in the integrand results in zero because it is constant. The integration of the rest is
\[
Q = \left[ \frac{1}{4\mu} \frac{\partial P}{\partial z} \right] \int_{r_i}^{r_o} \left[ r^2 + \ln \left( \frac{r_o}{r_i} \right) \left( r_o^2 - r_i^2 \right) \ln r \right] \, 2\pi r \, dr
\tag{104}
\]
The first integration of the first part of the second square bracket, \(1/r^3\), is \((1/4)\ln \left( \frac{r_o}{r_i} \right)\). The second part, of the second square bracket, \(-a \times r \ln r\) can be done by parts to be
Applying all these "techniques" to equation (104) results in

\[
Q = \left[ \frac{\pi}{2\mu} \left( \frac{\partial P}{\partial z} - \frac{\partial P}{\partial \theta} \right) \right]
\left[ \left( \frac{r_o}{r_i} \right)^{\frac{1}{4}} - \frac{1}{4} \ln \left( \frac{r_o}{r_i} \right) \right]
\]
The averaged velocity is obtained by dividing flow rate by the area \( Q/A \).

\[
U_{\text{ave}} = \frac{Q}{\pi \left( r_o^2 - r_i^2 \right)} \tag{105}
\]

in which the identity of \((a^4-b^4)/(a^2-b^2)\) is \(b^2+a^2\) and hence

\[
\begin{multline*}
U_{\text{ave}} = \left[ \frac{1}{2\mu} \frac{\partial P}{\partial z} \right] \\
+ \ln \left( \frac{r_o}{r_i} \right)
\end{multline*}
\]
Example 8.8

For the contraflow velocity profile, at what radius the maximum velocity obtained. Draw the maximum velocity location as a function of the ratio $\frac{r_i}{r_o}$.

The next example deals with the gravity as body force in two dimensional flow. This problem study by Nusselt which developed the basics equations. This problem is related to many industrial process and is fundamental in understanding many industrial processes. Furthermore, this analysis is a building block for heat and mass transfer understanding.

Example 8.9

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Fig. 8.20 Mass flow due to temperature difference for example
In many situations in nature and many industrial processes liquid flows downstream on inclined plate at $\theta$ as shown in Figure 8.20. For this example, assume that the gas density is zero (located outside the liquid domain). Assume that "scale" is large enough so that the "no slip" condition prevail at the plate (bottom). For simplicity, assume that the flow is two dimensional. Assume that the flow obtains a steady state after some length (and the acceleration vanished). The dominate force is the gravity. Write the governing equations for this situation. Calculate the velocity profile. Assume that the flow is one dimensional in the $(x)$ direction.

Solution 8.9

This problem is suitable to Cartesian coordinates in which $(x)$ coordinate is pointed in the flow direction and $(y)$ perpendicular to flow direction (depicted in Figure 8.20). For this system, the gravity in the $(x)$ direction is $(g\sin \theta)$ while the direction of $(y)$ the gravity is $(g\cos \theta)$. The governing in the $(x)$ direction is

\[
\rho \left( \overbrace{\dfrac{\partial U_x}{\partial t}}^{= f(t)} + \right. 
\overbrace{U_x \dfrac{\partial U_x}{\partial x}}^{= 0} + \overbrace{U_y}^{= 0} \dfrac{\partial U_x}{\partial y} + \left. \overbrace{U_z}^{-0} \dfrac{\partial U_x}{\partial z} \right) = \
- \overbrace{\dfrac{\partial P}{\partial x}}^{\sim 0} + \mu \left( \overbrace{\dfrac{\partial^2 U_x}{\partial x^2}}^{= 0} + \dfrac{\partial^2 U_x}{\partial y^2} + \overbrace{\dfrac{\partial^2 U_x}{\partial z^2}}^{= 0} \right) + \rho \overbrace{g_x}^{g \sin \theta}
\tag{106}
\]

The first term of the acceleration is zero because the flow is in a steady state. The first term of the convective acceleration is zero under the assumption of this example flow is fully developed and hence not a function of $(x)$ (nothing to be "improved"). The second and the third terms in the convective acceleration are zero because the velocity at that direction is zero ($U_y=U_z=0$). The pressure is almost constant along the $(x)$ coordinate. As it will be shown later, the pressure loss in the gas phase (mostly air) is negligible. Hence the pressure at the gas phase is almost constant hence the pressure at the interface in the liquid is constant. The surface has no curvature and hence the pressure at liquid side similar to the gas phase and the only change in liquid is in the $(y)$ direction. Fully developed flow means that the first term of the velocity Laplacian is zero ($\dfrac{\partial U_x}{\partial x}$ quiv 0). The last term of the velocity Laplacian is zero because no velocity in the $(z)$ direction. Thus, equation (106) is reduced to

\[
0 = \mu \overbrace{\dfrac{\partial^2 U_x}{\partial y^2}}^{(g \sin \theta)} + \rho \overbrace{g \sin \theta} \tag{107}
\]

With boundary condition of "no slip" at the bottom because the large scale and steady state

\[
U_x \ (y=0) = 0 \tag{108}
\]
The boundary at the interface is simplified to be
\[
\left. \frac{\partial U_x}{\partial y} \right|_{y=0} = \tau_{\text{air}} \left( \sim 0 \right) \tag{109}
\]
If there is additional requirement, such a specific velocity at the surface, the governing equation cannot be sufficient from the mathematical point of view. Integration of equation (107) yields
\[
\frac{\partial U_x}{\partial y} = \frac{\rho}{\mu} g \sin \theta \, y + c_1 \tag{110}
\]
The integration constant can be obtain by applying the condition (109)
\[
\tau_{\text{air}} = \mu \left. \frac{\partial U_x}{\partial y} \right|_h = - \rho g \sin \theta \, h + c_1 \mu \tag{111}
\]
Solving for \(c_1\) results in
\[
c_1 = \frac{\tau_{\text{air}}}{\mu} + \frac{1}{\nu} g \sin \theta \, h \tag{112}
\]
The second integration applying the second boundary condition yields \(c_2=0\) results in
\[
U_x = \frac{g \sin \theta}{\nu} \left( 2h y - y^2 \right) - \frac{\tau_{\text{air}}}{\mu} \tag{113}
\]
When the shear stress caused by the air is neglected, the velocity profile is
\[
U_x = \frac{g \sin \theta}{\nu} \left( 2h y - y^2 \right) \tag{114}
\]
The flow rate per unit width is
\[
\frac{Q}{W} = \int_A U_x \, dA = \int_0^h \left( \frac{g \sin \theta}{\nu} \left( 2h y - y^2 \right) - \frac{\tau_{\text{air}}}{\mu} \right) \, dy \tag{115}
\]
Where \(W\) here is the width into the page of the flow. Which results in
\[
\frac{Q}{W} = \frac{g \sin \theta}{\nu} \frac{2h^3}{3} - \frac{\tau_{\text{air}} h}{\mu} \tag{116}
\]
The average velocity is then

\[
\overline{U_x} = \frac{\frac{Q}{W}}{h} = \frac{g \sin \theta}{\nu} \frac{2h^2}{3} - \frac{\tau_{air}}{\mu} \tag{117}\]

Note the shear stress at the interface can be positive or negative and hence can increase or decrease the flow rate and the averaged velocity.

In the following example the issue of driving force of the flow through curved interface is examined. The flow in the kerosene lamp depends on the surface tension. The flow surface is curved and thus pressure is not equal on both sides of the interface.

Example 8.10

A simplified flow version the kerosene lump is of liquid moving up on a solid core. Assume that radius of the liquid and solid core are given and the flow is at steady state. Calculate the minimum shear stress that required to operate the lump (alternatively, the maximum height).

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