10.4.1: Complex Potential and Complex Velocity

The definition of Cauchy–Riemann equations can lead to the definition of the complex potential \( F(z) \) as following

\[
F(z) = \phi(x,y) + i\psi(x,y) \tag{1}
\]

where \( z = x + i\,y \). This definition based on the hope that \( F \) is differentiable and continuous in other words analytical. In that case a derivative with respect to \( z \) when \( z \) is real number is

\[
\frac{dF}{dz} = \frac{dF}{dx} = \frac{d\phi}{dx} + i\,\frac{d\psi}{dx} \tag{2}
\]

On the other hand, the derivative with respect to the \( \bar{z} \) that occurs when \( \bar{z} \) is pure imaginary number then

\[
\frac{dF}{d\bar{z}} = \frac{1}{i}\frac{dF}{dy} = -i\,\frac{dF}{dy} = -\frac{d\phi}{dy} + \frac{d\psi}{dy} \tag{3}
\]

Equations (2) and (3) show that the derivative with respect to \( \bar{z} \) depends on the orientation of \( \bar{z} \). It is desired that the derivative with respect to \( \bar{z} \) will be independent of the orientation. Hence, the requirement is that the result in both equations must be identical. Hence,

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{4}
\]
In fact, the reverse also can be proved that if the Cauchy–Riemann equations condition exists it implies that the complex derivative also must be exist. Hence, using the complex number guarantees that the Laplacian of the stream function and the potential function must be satisfied (why?). While this method cannot be generalized three dimensions it provides good education purposes and benefits for specific cases. One major advantage of this method is the complex number technique can be used without the need to solve differential equation. The derivative of the \(F\) is independent of the orientation of the \(z\)

\[
W(z) = \frac{dF}{dz} \tag{5}
\]

This also can be defined regardless as the direction as

\[
W(z) = \frac{dF}{dx} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \tag{6}
\]

Using the definition that were used for the potential and the stream functions, one can obtain that

\[
\frac{dF}{dz} = U_x - i \, U_y \tag{7}
\]

The characteristic complex number when multiplied by the conjugate, the results in a real number (hence can be view as scalar) such as

\[
\overline{W} = \left(U_x - i \, U_y \right) \left(U_x + i \, U_y \right) = \{U_x\}^2 + \{U_y\}^2 \tag{8}
\]

In Bernoulli’s equation the summation of the squares appear and so in equation (??). Hence, this multiplication of the complex velocity by its conjugate needs velocity for relationship of pressure–velocity. The complex numbers sometimes are easier to handle using polar coordinates in such case like finding roots etc. From the Figure the following geometrical transformation can be written

\[
U_x = U_r \cos \theta - U_{\theta} \sin \theta \tag{9}
\]

and

\[
U_x = U_r \sin \theta + U_{\theta} \sin \theta \tag{10}
\]

Using the above expression in the complex velocity yields
\[
W = (U_r \cos \theta - U_\theta \sin \theta) - i(U_r \sin \theta + U_\theta \cos \theta) \tag{11}
\]
Combining the \((r)\) and \((\theta)\) component separately
\[
W = U_r \left( \cos \theta - i \sin \theta \right) - U_\theta \left( \cos \theta - i \sin \theta \right) \tag{12}
\]
It can be noticed the Euler identity can be used in this case to express the terms that, are multiplying the velocity and since they are similar to obtain
\[
W = (U_r - i U_\theta) e^{-i \theta} \tag{13}
\]

**Uniform Flow**

The uniform flow is revisited here with a connection to the complex numbers presentation. In the previous section, the uniform flow was present as the flow from the left to right. Here, this presentation will be expanded. The connection between the mathematical presentation to the physical flow is weak at best and experience is required. One can consider the flow that described by the function

\[
F(z) = c \cdot z = c \cdot (x + i \cdot y) \tag{14}
\]
The the complex flow is
\[
W = \frac{dF}{dz} = c \tag{15}
\]
The complex velocity was found to be represented as
\[
W = c = U_x - i \cdot U_y \tag{16}
\]
There are three extreme cases that need to be examined. The first case is when \(|c|\) is a real number. In that case, it requires that \(|U_x = c|\) which is exactly the case that was presented earlier. The case the constant is imaginary resulting in
\begin{align}
U_x - iU_y &= -i c \tag{17} \\
\text{When it was chosen that the constant value is negative it yields} \\
U_y &= c \tag{18}
\end{align}

This kind of flow is when the direction is upward and was not discussed in the standard presentation earlier. The third case, the constant is a complex number. In that case, the complex number is present in either polar coordinate for convenience or in Cartesian coordinate to be as

\begin{align}
F(z) &= c e^{-i\theta} z \tag{20} \\
The complex velocity will be then \\
W(z) &= c \cos \theta - i c \sin \theta \tag{21} \\
Hence the component of the velocity are \\
\begin{array}{c}
U_x = c \cos \theta \\
U_y = c \sin \theta 
\end{array} \tag{22}
\end{align}

This flow is the generalized uniform flow where the flow is in arbitrary angle with the coordinates. In general the uniform flow is described in two-dimensional field as

\begin{align}
F(z) &= U_0 e^{-i\theta} z \tag{23} \\
\text{This flow contains two extremes cases discussed earlier horizontal and vertical flow.}
\end{align}

\section*{Flow in a Sector}

The uniform flow presentation seem to be just repeat of what was done in the presentation without the complex numbers. In sector flow is an example where the complex number presentation starts to shine. The sector flow is referred to as a flow in sector. Sector is a flow in opening with specific angle. The potential is defined as
\[ F(z) = U_0 \, z^n \tag{24} \]
where \((n \geq 1)\) the relationship between the \((n)\) and opening angle will be established in this development. The polar represented is used in this derivations as \((z=r, e^{i\theta})\) and substituting into equation (24) provides

\[ F(z) = U_0 \, r^n \cos(n\theta) + i \, U_0 \, r^n \sin(n\theta) \tag{25} \]

The potential function is

\[ \phi = U_0 \, r^n \cos(n\theta) \tag{26} \]

and the stream function is

\[ \psi = U_0 \, r^n \sin(n\theta) \tag{27} \]

The stream function is zero in two extreme cases: one when the \((\theta=0)\) and two when \((\theta = \pi/n)\). The stream line where \((|\psi|=0)\) are radial lines at the angles and \((|\theta|=0)\) and \((|\theta|=\pi/n)\). The zone between these two line the streamline are defined by the equation of \((|\psi = U_0 \, r^n \sin(n\theta))\). The complex velocity can be defined as the velocity along these lines and is

\[ W(z) = n \, U_0 \, z^{n-1} = n \, U_0 \, r^{n-1} \, e^{i \, (n-1)\theta} = n \, U_0 \, r^{n-1} \cos(n\theta) + i \, n \, U_0 \, r^{n-1} \sin(n\theta) \, e^{i\theta} \tag{28} \]

Thus the velocity components are

\[ U_r = n \, U_0 \, r^{n-1} \cos(n\theta) \tag{29} \]

and

\[ U_\theta = n \, U_0 \, r^{n-1} \sin(n\theta) \tag{30} \]

It can be observed that the radial velocity is positive in the range of \((0 < \theta < \frac{\pi}{2 \, n})\) while it is negative in the range \((\frac{\pi}{2 \, n} < \theta < \frac{\pi}{n})\). The tangential velocity is negative in the \((0 < \theta < \frac{\pi}{2 \, n})\) while it is positive in the range \((\frac{\pi}{2 \, n} < \theta < \frac{\pi}{n})\). In the above discussion it was
established the relationship between the sector angle and the power \( n \). For \( n \) the flow became uniform and increased of the value of the power, \( n \) reduce the sector. For example if \( n=2 \) the flow is in a right angle sector. Generally the potential of shape corner is given by

\[
F(z) = U_0 \, z^n \tag{31}
\]

Flow around a Sharp Edge

It can be observed that when \( n<1 \) the angle is larger then \( \pi \) this case of flow around sharp corner. This kind of flow creates a significant acceleration that will be dealt in some length in compressible flow under the chapter of Prandtl-Meyer Flow. Here it is assumed that the flow is ideal and there is continuation in the flow and large accelerations are possible. There is a specific situation where there is a turn around a a flat plate. In this extreme case is when the value of \( n<0.5 \). In that case, the flow turn around the \( 2\pi \) angle. In that extreme case the complex potential function is

\[
F(z) = c\sqrt{z} \tag{32}
\]

If the value of \( c \) is taken as real the angle must be limited within the standard \( 360^\circ \) and the explicit potential in polar coordinates is

\[
F(z) = c\sqrt{r}\,e^{0.5\,i\theta} \tag{33}
\]

The the potential function is

\[
\phi = c\sqrt{r}\cos\left(\frac{\theta}{2}\right) \tag{34}
\]

The stream function is

\[
\psi = c\sqrt{r}\sin\left(\frac{\theta}{2}\right) \tag{35}
\]

The streamlines are along the part the \( \sin \) zero which occur at \( \theta=0 \) and \( \theta=2\pi \).

Contributors

- Dr. Genick Bar-Meir. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or later or Potto license.